Math 46 - Fall 2021 - Discussion section 002 and 005
Instructor: Prof. Feng Xu
Homework due Oct 22

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1 2.4. Transformation of Non Linear Equations into Separable Equations

Exercise 1. A Bernoulli differential equation is one of the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n.$$

Observe that, if $n = 0$ or 1, the Bernoulli equation is linear. For other values of n, the substitution $u = y^{1-n}$ transforms the Bernoulli equation into the linear equation

$$\frac{du}{dx} + (1-n)P(x)u = (1-n)Q(x).$$

Use an appropriate substitution to solve the equation

$$xy' + y = 9xy^2,$$

and find the solution that satisfies $y(1) = -5$.

Exercise 2. A Bernoulli differential equation is one of the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n \qquad (*)$$

Observe that, if $n = 0$ or 1, the Bernoulli equation is linear. For other values of n, the substitution $u = y^{1-n}$ transforms the Bernoulli equation into the linear equation

$$\frac{du}{dx} + (1-n)P(x)u = (1-n)Q(x).$$

Consider the initial value problem

$$xy' + y = -4xy^2, \quad y(1) = -5.$$

(a) This differential equation can be written in the form $(*)$ with $P(x) =?$, $Q(x) =?$, and $n =?$

(b) The substitution $u =?$ will transform it into the linear equation $\frac{du}{dx} + ?u = ?$

(c) Using the substitution in part (b), we rewrite the initial condition in terms of x and u: $u(1) =?$

(d) Now solve the linear equation in part (b), and find the solution that satisfies the initial condition in part (c). $u(x) =?$

(e) Finally, solve for y.

$y(x) =$ ________________________.

Exercise 3. A Bernoulli differential equation is one of the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n.$$

Observe that, if $n = 0$ or 1, the Bernoulli equation is linear. For other values of n, the substitution $u = y^{1-n}$ transforms the Bernoulli equation into the linear equation

$$\frac{du}{dx} + (1-n)P(x)u = (1-n)Q(x).$$

Use an appropriate substitution to solve the equation

$$\boxed{y' - \frac{6}{x}y = \frac{y^4}{x^{16}},}$$

and find the solution that satisfies $y(1) = 1$.

$u = y^{1-4} = y^{-3} = \frac{1}{y^3}$

$n = 4$: $\boxed{u' = -3y^{-4}y'} = \frac{-3y'}{y^4}$

$\frac{y'}{y^4} - \frac{6}{x}\frac{1}{y^3} = \frac{1}{x^{16}}$

$\frac{u'}{-3} - \frac{6}{x}u = \frac{1}{x^{16}}$

$\frac{u'}{-3} - \frac{6}{x}u = 0 \Rightarrow \frac{u'}{u} = \cancel{\frac{-3}{6x}} = \frac{-18}{x}$

$\frac{u'}{-3} = \frac{6}{x}u \Rightarrow \frac{u'}{-3u} = \frac{6}{x} \Rightarrow \frac{u'}{u} = \frac{-18}{x}$

$\ln u = -18\ln x = \ln x^{-18} \Rightarrow u_1 = x^{-18}$

$u = v\cdot u_1 = v\cdot x^{-18} \Rightarrow u' = v'x^{-18} - 18vx^{-19}$

$\frac{u'}{-3} = \frac{v'x^{-18}}{-3} + 6vx^{-19}$

$\frac{u'}{-3} - \frac{6}{x}u = \frac{1}{x^{16}}$

$\frac{v'x^{-18}}{-3} + \underbrace{6vx^{-19}} - \underbrace{\frac{6}{x}\cdot v\cdot x^{-18}} = \frac{1}{x^{16}}$

$\frac{v'x^{-18}}{-3} = \frac{1}{x^{16}} \Rightarrow v' = \frac{-3x^{18}}{x^{16}} = -3x^2$

$\Rightarrow v = \int -3x^2 = -x^3 + C$

$u = (-x^3 + C)\cdot x^{-18}$

$\frac{1}{y^3} = (-x^3 + C)\cdot\frac{1}{x^{18}}$: $y(1) = 1$

$\frac{1}{1} = (-1 + C)\cdot\frac{1}{1} \Rightarrow C = 2$

$\frac{1}{y^3} = (-x^3 + 2)\cdot\frac{1}{x^{18}}$

$y^3 = \frac{x^{18}}{-x^3 + 2}$

$y = \frac{x^6}{\sqrt[3]{-x^3 + 2}}$

2 Exact

Exercise 1. Use the "mixed partials" check to see if the following differential equation is exact.

If it is exact find a function F(x,y) whose differential, $dF(x,y)$ is the left hand side of the differential equation. That is, level curves $F(x,y) = C$ are solutions to the differential equation:

$$(-1x^4 + 2y)dx + (2x + 3y^2)dy = 0$$

First:

$M_y(x,y) =?$, and $N_x(x,y) =?$

If the equation is not exact, enter *not exact,* otherwise enter in $F(x,y)$ here ______________

Exercise 2. Use the "mixed partials" check to see if the following differential equation is exact.
If it is exact find a function F(x,y) whose differential, $dF(x, y)$ is the left hand side of the differential equation. That is, level curves $F(x, y) = C$ are solutions to the differential equation:

$$\overset{M}{(2xy^2 + 3y)}dx + \overset{N}{(2x^2y + 3x)}dy = 0$$

First: $M_y(x, y) = \underline{\hspace{3cm}}$, and $N_x(x, y) = \underline{\hspace{3cm}}$.
If the equation is not exact, enter *not exact,* otherwise enter in $F(x, y)$ here $\underline{\hspace{3cm}}$

Exercise 3. Use the "mixed partials" check to see if the following differential equation is exact.

If it is exact find a function $F(x, y)$ whose differential, $dF(x, y)$ gives the differential equation. That is, level curves $F(x, y) = C$ are solutions to the differential equation:

$$\boxed{\frac{dy}{dx} = \frac{x^3 - y}{x + 4y^4}} \qquad (x + 4y^4)\,dy = (x^3 - y)\,dx$$

First rewrite as

$$M(x, y)\,dx + N(x, y)\,dy = 0 \qquad \underbrace{(x^3 - y)}_{M}dx + \underbrace{(-x - 4y^4)}_{N}dy = 0$$

where $M(x, y) =$ ____________ ,

and $N(x, y) =$ ____________.

If the equation is not exact, enter in *not exact*, otherwise enter in $F(x, y)$ as the solution of the differential equation here ________________ $= C$.

$$\Rightarrow M_y = \frac{d}{dy}(x^3 - y) = 0 - 1 = -1 \qquad M_y = N_x$$

$$N_x = \frac{d}{dx}(-x - 4y^4) = -1 + 0 = -1 \qquad \Rightarrow \text{exact.}$$

$$F(x, y) = \int M\,dx = \int (x^3 - y)\,dx = \frac{x^4}{4} - yx + g(y)$$

$$\frac{d}{dy}F(x, y) = \frac{d}{dy}\left(\frac{x^4}{4} - yx + g(y)\right) = \boxed{0 - x + g'(y) = N = -x - 4y^4}$$

$$g'(y) = -4y^4 \Rightarrow g(y) = \frac{-4y^5}{5} + C$$

$$F(x, y) = \underbrace{\frac{x^4}{4} - yx - \frac{4y^5}{5}}_{} + C$$

Exercise 4. Use the "mixed partials" check to see if the following differential equation is exact.

If it is exact find a function $F(x, y)$ whose differential, $dF(x, y)$ is the left hand side of the differential equation.

That is, level curves $F(x, y) = C$ are solutions to the differential equation

$$(-4e^x \sin(y) + 2y)dx + (2x + 4e^x \cos(y))dy = 0$$

First, if this equation has the form $M(x, y)dx + N(x, y)dy = 0$:

$M_y(x, y) =$ ______________ , and $N_x(x, y) =$ ______________.

If the equation is not exact, enter *not exact,* otherwise enter in $F(x, y)$ here ______________

Exercise 5. The differential equation

$$y - 3y^2 = \left(y^6 + x\right) y'$$

can be written in differential form:

$$M(x, y)\, dx + N(x, y)\, dy = 0$$

where
$M(x, y) =$ ______________ , and $N(x, y) =$ ______________.

The term $M(x, y)\, dx + N(x, y)\, dy$ becomes an exact differential if the left hand side above is divided by y^2. Integrating that new equation, the solution of the differential equation is _____________________ $= C$.