Math 46 - Fall 2021 - Discussion section 002 and 005
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1 5.7: Variation of Parameters

Exercise 1. Find a particular solution to

$$\boxed{1 = P_0 \; y''} + 2y' + y = \frac{17.5e^{-t}}{t^2+1} \rightarrow F.$$

Answer: $y_p = 17.5 * e^{(-1*t)}(-\ln(1+t^2)/2 + t * a * \tan(t)) + a * e^{(-1*t)} + b * t * e^{(-1*t)}$

Step 1: Start with homogeneous: $y'' + 2y' + y = 0$

$r^2 + 2r + 1 = 0$

$(r+1)^2 = 0$, $\boxed{r = -1}$

$$\boxed{y_1 = e^{-1x}, \quad y_2 = x \cdot e^{-1x}}$$

Step 2: $\boxed{y = u_1 \cdot y_1 + u_2 \cdot y_2}$ Goal: Find u_1 and u_2

(1): $u_1' \cdot y_1 + u_2' \cdot y_2 = 0$

(2): $u_1' \cdot y_1' + u_2' \cdot y_2' = \dfrac{F}{P_0} = \dfrac{17.5e^{-t}}{t^2+1}$

(1): $u_1' \cdot e^{-x} + u_2' \cdot xe^{-x} = 0$

+

(2): $\underbrace{u_1' \cdot (-e^{-x})} + u_2'(e^{-x} \underbrace{- xe^{-x}}) = \dfrac{17.5e^{-t}}{t^2+1}$

①+②:

$$u_2' e^{-x} = \frac{17.5e^{-t}}{t^2+1}$$

$$u_2' e^{-t} = \frac{17.5e^{-t}}{t^2+1}$$

$$\boxed{u_2' = \frac{17.5}{t^2+1}}$$

$$u_1' \cdot e^{-t} + \frac{17.5}{t^2+1} \cdot t \cdot e^{-t} = 0$$

$$u_1' + \frac{17.5}{t^2+1} \cdot t = 0$$

$$\boxed{u_1' = -\frac{17.5t}{t^2+1}}$$

$$u_2 = \int u_2' = \int \frac{17.5}{t^2+1}\,dt \qquad \tan^{-1} t$$

$$= 17.5 \int \frac{1}{t^2+1}\,dt = 17.5(\arctan t + C_1)$$

$$u_1 = -17.5 \int \frac{t}{t^2+1}\,dt \qquad v = t^2+1,\quad dv = 2t\,dt$$

$$= -17.5 \int \frac{dv}{2v}$$

$$= -17.5 \cdot \frac{1}{2} \ln|v| = -\frac{17.5}{2} \ln(t^2+1)$$

Exercise 2. Solve the following differential equation by variation of parameters. Fully evaluate all integrals.

$$y'' + 9y = \sec(3x).$$

Find the most general solution to the associated homogeneous differential equation. Use c_1 and c_2 in your answer to denote arbitrary constants.
Answer: $y_h = c_1 * \cos(3 * x) + c_2 * \sin(3 * x)$

Find a particular solution to the nonhomogeneous differential equation $y'' + 9y = \sec(3x)$

Answer: $\boxed{a * \cos(3 * x)} + \boxed{b * \sin(3 * x)} + \boxed{1/3 * x * \sin(3 * x)} + \boxed{1/9 * \cos(3 * x) * \ln(|\cos(3 * x)|)}$

Find the most general solution to the original nonhomogeneous differential equation.

Answer: $c_1 * \cos(3 * x) + c_2 * \sin(3 * x) + 1/3 * x * \sin(3 * x) + 1/9 * \cos(3 * x) * \ln(|\cos(3 * x)|)$

$u_1 \cos 3x + u_2 \sin 3x$ Find u_1 and u_2

$u_1 = \left[\frac{1}{9} \ln|\cos 3x| + a\right]$

$u_2 = \left[\frac{1}{3}x + b\right]$ $\quad u_2 \cdot \sin 3x$

Exercise 3. In this exercise you will solve the initial value problem

$$y'' - 18y' + 81y = \frac{e^{-9x}}{1+x^2}, \quad y(0) = 8, \quad y'(0) = 4.$$

Let C_1 and C_2 be arbitrary constants. The general solution to the related homogeneous differential equation $y'' - 18y' + 81y = 0$ is the function $y_h(x) = C_1\, y_1(x) + C_2\, y_2(x)$.

NOTE: The order in which you enter the answers is important;

Answer: $C_1 e^{9x} + C_2 * x * e^{9x}$ $\qquad y_1 = e^{9x}, \quad y_2 = x e^{9x}$

The particular solution $y_p(x)$ to the differential equation $y'' + 18y' + 81y = \frac{e^{-9x}}{1+x^2}$ is of the form $y_p(x) = y_1(x)\, u_1(x) + y_2(x)\, u_2(x)$ where $u_1'(x) =$? and $u_2'(x) =$?.

Answer: $\boxed{u_1'(x) = -\frac{xe^{-27x}}{1+x^2}}$ and $\boxed{u_2'(x) = \frac{e^{-18x}}{1+x^2}}$

The most general solution is: $y = C_1 \underbrace{e^{9x}}_{y_1} \underbrace{\int_0^x -\frac{te^{-27t}}{1+t^2}dt}_{\int_0^x u_1'(t)dt} + C_2 \underbrace{xe^{9x}}_{y_2} \underbrace{\int_0^x \frac{e^{-18t}}{1+t^2}dt}_{\int_0^x u_2'(t)dt}$