Math 46 - Fall 2021 - Discussion section 002 and 005
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1 5.7: Variation of Parameters

Exercise 1. Find a particular solution to

$$y'' + 2y' + y = \frac{17.5e^{-t}}{t^2 + 1}\ .$$

Answer: $y_p = 17.5 * e^{(-1*t)}(-\ln(1 + t^2)/2 + t * a * \tan(t)) + a * e^{(-1*t)} + b * t * e^{(-1*t)}$

Exercise 2. Solve the following differential equation by variation of parameters. Fully evaluate all integrals.

$$y'' + 9y = \sec(3x).$$

Find the most general solution to the associated homogeneous differential equation. Use c_1 and c_2 in your answer to denote arbitrary constants.
Answer: $y_h = c_1 * \cos(3 * x) + c_2 * \sin(3 * x)$

Find a particular solution to the nonhomogeneous differential equation $y'' + 9y = \sec(3x)$

Answer: $a * \cos(3 * x) + b * \sin(3 * x) + 1/3 * x * \sin(3 * x) + 1/9 * \cos(3 * x) * \ln(|\cos(3 * x)|)$

Find the most general solution to the original nonhomogeneous differential equation.

Answer:$c_1 * \cos(3 * x) + c_2 * \sin(3 * x) + 1/3 * x * \sin(3 * x) + 1/9 * \cos(3 * x) * \ln(|\cos(3 * x)|)$

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Exercise 3. In this exercise you will solve the initial value problem

$$P_0 \quad y'' - 18y' + 81y = \frac{e^{-9x}}{1+x^2}, \quad y(0) = 8, \quad y'(0) = 4.$$

Let C_1 and C_2 be arbitrary constants. The general solution to the related homogeneous differential equation $y'' - 18y' + 81y = 0$ is the function $y_h(x) = C_1\, y_1(x) + C_2\, y_2(x)$.

NOTE: The order in which you enter the answers is important;

Answer: $\boxed{C_1 e^{9x} + C_2 * x * e^{9x}}$

The particular solution $y_p(x)$ to the differential equation $y'' + 18y' + 81y = \frac{e^{-9x}}{1+x^2}$ is of the form $y_p(x) = y_1(x)\, u_1(x) + y_2(x)\, u_2(x)$ where $u_1'(x) =?$ and $u_2'(x) =?$.

Answer: $u_1'(x) = -\frac{xe^{-27x}}{1+x^2}$ and $u_2'(x) = \frac{e^{-18x}}{1+x^2}$

The most general solution is: $y = C_1 e^{9x} \int_0^x -\frac{te^{-27t}}{1+t^2}dt + C_2 x e^{9x} \int_0^x \frac{e^{-18t}}{1+t^2}dt$

$u'(x)$ $\quad u(x) \overset{?}{=} \int_0^x u'(t)\,dt$

Fundamental Theorem of Calculus.

Step 1: $y'' - 18y' + 81 = 0 : r^2 - 18r + 81 = 0 : (r-9)^2 = 0 : \boxed{r = 9}$

$y_1 = e^{9x}, \quad y_2 = xe^{9x} \quad \boxed{: y_h = C_1 y_1 + C_2 y_2}$

Step 2: $\boxed{y_p = u_1 y_1 + u_2 y_2}$ Goal: find u_1 and u_2.

① : $u_1' y_1 + u_2' y_2 = 0$

② : $u_1' \cdot y_1' + u_2' y_2' = \frac{F}{P_0} = \frac{e^{-9x}}{1+x^2}$

$\boxed{P_0 : \text{coefficient of } y''}$ "F: RHS"

$$\boxed{u_1' = \frac{-x e^{-9x}}{e^{9x}(1+x^2)}}$$

$$y_p = \underbrace{C_1 e^{9x}}_{C_1 y_1} \cdot \underbrace{\int_0^x \frac{-t e^{-9t}}{e^{9t}(1+t^2)}\,dt}_{u_1} + \underbrace{C_2 x e^{9x}}_{C_2 y_2} \cdot \underbrace{\int_0^x \frac{-e^{-9t}}{e^{9t}(1+t^2)}\,dt}_{u_2}$$

(1) $u_1' \cdot e^{9x} + u_2' \cdot x e^{9x} = 0$

(2) $u_1' \cdot (9e^{9x}) + u_2' (e^{9x} + x \cdot 9 e^{9x}) = \dfrac{e^{-9x}}{1+x^2}$

(1) multiply by 9:

$$9u_1' e^{9x} + u_2' 9x e^{9x} = 0 \quad (3)$$

Take (2) substract (3):

$$u_2' e^{9x} = \frac{e^{-9x}}{1+x^2} \Rightarrow \boxed{u_2' = \frac{e^{-9x}}{e^{9x}(1+x^2)}}$$

Plug into 1: $u_1' \cancel{e^{9x}} + \dfrac{e^{-9x}}{e^{9x}(1+x^2)} \cdot x \cancel{e^{9x}} = 0$

$$\boxed{u_1' = \frac{-x e^{-9x}}{e^{9x}(1+x^2)}}$$

$$y_p = \underbrace{C_1 e^{9x}}_{C_1 y_1} \cdot \underbrace{\int_0^x \frac{-t e^{-9t}}{e^{9t}(1+t^2)}\,dt}_{u_1} + \underbrace{C_2 x e^{9x}}_{C_2 y_2} \cdot \underbrace{\int_0^x \frac{-e^{-9t}}{e^{9t}(1+t^2)}\,dt}_{u_2}$$

$$(1)\quad u_1' \cdot e^{9x} + u_2' \cdot x e^{9x} = 0$$

$$(2)\quad u_1' \cdot (9e^{9x}) + u_2'\left(e^{9x} + x \cdot 9e^{9x}\right) = \frac{e^{-9x}}{1+x^2}$$

(1) multiply by 9:

$$9u_1' e^{9x} + u_2' \, 9x e^{9x} = 0 \quad (3)$$

Take (2) substract (3):

$$u_2' e^{9x} = \frac{e^{-9x}}{1+x^2} \Rightarrow \boxed{u_2' = \frac{e^{-9x}}{e^{9x}(1+x^2)}}$$

Plug into 1: $u_1' \cancel{e^{9x}} + \frac{e^{-9x}}{e^{9x}(1+x^2)} \cdot x \cancel{e^{9x}} = 0$

$y(0) = 8, \; y'(0) = 4.$

$x = 0, \; y = 8:$

$8 = c_1 \cdot 1 \cdot 1 + \cancel{c_2 \cdot 0}: \; \boxed{c_1 = 8}$

$$y_h = \underbrace{8e^{9x}} + c_2 x e^{9x}$$

$$y_h' = (72e^{9x} + 8e^{9x})' + c_2 e^{9x} + c_2 x 9 e^{9x}$$

$x = 0, \; y_h' = 4:$

$4 = 72 + 8 + c_2 + \cancel{c_2 \cdot 0}$

$$\boxed{c_2 = -76}$$

$$\boxed{u_1' = \frac{-xe^{-9x}}{e^{9x}(1+x^2)}}$$

$$y_p = \underbrace{C_1 e^{9x}}_{C_1 y_1} \cdot \underbrace{\int_0^x \frac{-te^{-9t}}{e^{9t}(1+t^2)}\,dt}_{u_1} + \underbrace{C_2 xe^{9x}}_{C_2 y_2} \cdot \underbrace{\int_0^x \frac{-e^{-9t}}{e^{9t}(1+t^2)}\,dt}_{u_2}$$

(1) $u_1' \cdot e^{9x} + u_2' \cdot xe^{9x} = 0$

(2) $u_1' \cdot (9e^{9x}) + u_2'(e^{9x} + x \cdot 9e^{9x}) = \dfrac{e^{-9x}}{1+x^2}$

(1) multiply by 9:

$9u_1' e^{9x} + u_2' 9xe^{9x} = 0 \quad (3)$

Take (2) substract (3):

$$u_2' e^{9x} = \frac{e^{-9x}}{1+x^2} \Rightarrow \boxed{u_2' = \frac{e^{-9x}}{e^{9x}(1+x^2)}}$$

Plug into 1: $u_1' \cancel{e^{9x}} + \dfrac{e^{-9x}}{e^{9x}(1+x^2)} \cdot x\cancel{e^{9x}} = 0$