Math 46 - Fall 2021 - Discussion section 002 and 005
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1 5.4 and 5.5: Method of Undetermined Coefficients

Exercise 1. Use the method of undetermined coefficients to solve the following differential equation:

$$\boxed{y'' + y' = 4x} \to \text{“}4\boxed{x}\text{”}$$

Answer: $y = \underbrace{2x^2 - 4x}_{y_p} + C_1 \times 1 + C_2 \times e^{-x}$

Step 1: Start with homogeneous: $y'' + y' = 0$

$$r^2 + r = 0$$
$$r(r+1) = 0$$
$$\boxed{r = 0, \quad r = -1}$$
$$y = C_1 e^{0x} + C_2 e^{-1x}$$
$$= C_1 \cdot 1 + C_2 \cdot e^{-x}$$

Step 2: Find $y_p = x(\underbrace{Ax + b}_{\text{Polynomial}})$

$y_p = Ax^2 + bx \Rightarrow y_p' = 2Ax + b$

$y_p'' = 2A$

$y'' + y' = 4x$: $\underbrace{2A}_{y_p''} + \underbrace{2Ax + b}_{y_p'} = 4x$

So $2Ax = 4x$

$A = 2$

$2A + b = 0$ so $b = -2A = -2 \cdot 2 = -4$.

So $\boxed{y_p = x(2x-4) = 2x^2 - 4x}$

Exercise 2. Use the method of undetermined coefficients to solve the following differential equation:

$$y'' + 6y' + 9y = 2e^{-x} \longrightarrow -1$$

Answer: $y = \underbrace{\frac{1}{2} e^{-x}}_{y_p} + C_1 \times e^{-3x} + C_2 \times xe^{-3x}$

<u>Step 1</u>: Start with homogeneous: $y'' + 6y' + 9y = 0$

$r^2 + 6r + 9 = 0 : (r+3)^2 = 0$ $\boxed{r = -3}$

$$C_1 e^{-3x} + C_2 \cdot x \cdot e^{-3x}$$

<u>Step 2</u>: Find y_p: because $r = -3 \neq -1$:

$\underset{\text{guess}}{y_p = Ae^{-x}}$: $y_p' = -Ae^{-x}$

$y_p'' = Ae^{-x}$

$$y'' + 6y' + 9y = 2e^{-x}$$

$$Ae^{-x} + 6(-Ae^{-x}) + 9Ae^{-x} = 2e^{-x}$$

$$4Ae^{-x} = 2e^{-x} \Rightarrow \boxed{A = \frac{1}{2}}$$

$$\boxed{y_p = \frac{1}{2} e^{-x}}$$

Exercise 3. Use the method of undetermined coefficients to solve the following differential equation:

y_p

$$y'' + 4y = 4x$$

Answer: $y = x + C_1 \times \cos(2x) + C_2 \times \sin(2x)$

Step 1: Homogeneous: $y'' + 4y = 0$

α $\quad$ β

$r^2 + 4 = 0$: $r = \pm 2i = 0 \pm 2i$

$$y = C_1 \cdot e^{\alpha} \cdot \cos \beta x + C_2 \cdot e^{\alpha} \cdot \sin \beta x$$

$$\boxed{= C_1 \cos 2x + C_2 \sin 2x}$$

Step 2: Find y_p

$y_{p\,\text{guess}} = Ax$: $y_p' = A$, $y_p'' = 0$

$y'' + 4y = 4x$

$0 + 4Ax = 4x \Rightarrow A = 1$.

$$\boxed{y_p = x} \checkmark$$

Exercise 4. Use the method of undetermined coefficients to solve the following differential equation:

$$y'' + 6y' + 9y = 2\sin(x)$$

Answer: $y = .16\sin(x) - .12\cos(x) + C_1 \times e^{-3x} + C_2 \times xe^{-3x}$

Step 1: Homogeneous : $y'' + 6y' + 9y = 0$: $r^2 + 6r + 9 = 0$: $(r+3)^2 = 0$

$\boxed{r = -3}$

$$y = C_1 e^{-3x} + C_2 \cdot x \cdot e^{-3x}$$

Step 2: Solve for y_p : $y_p = A\sin x + B\cos x$

$y_p' = A\cos x - B\sin x$, $y_p'' = -A\sin x - B\cos x$

$$-A\sin x - B\cos x + 6(A\cos x - B\sin x) + 9(A\sin x + B\cos x) = 2\sin x$$

$$(8A - 6B)\sin x + (8B - 6A)\cos x = 2\sin x$$

$$8A - 6B = 2 \quad \Rightarrow \quad A = .16$$

$$8B - 6A = 0 \qquad\quad B = .12$$

$$y_p = .16\sin x - .12\cos x .$$

Exercise 5. Solve the following differential equation:

$$y'' + 4y = \sin^3 x \Rightarrow \boxed{\text{“}\sin 3x\text{”} = 3\sin x - 4\sin^3 x}$$

Answer: $y = \frac{1}{4}\sin(x) + \frac{1}{20}\sin(3x) + C_1 \times \cos(2x) + C_2 \times \sin(2x)$

$$y_p = A\sin x + B\sin 3x$$