Math 46 - Fall 2021 - Discussion section 002 and 005
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November 16, 2021

1 5.4 and 5.5: Method of Undetermined Coefficients

Exercise 1. Use the method of undetermined coefficients to solve the following differential equation:

$$y'' + y' = 4x$$

$\boxed{4x}$: power one : x

they give degree 1

Answer: $y = \underbrace{2x^2 - 4x}_{y_p} + \boxed{C_1 \times 1 + C_2 \times e^{-x}}$

Step 1: start with homogeneous: $y'' + y' = 0$

$r^2 + r = 0 \; : \; r(r+1) = 0$

$\boxed{r=0}$ or $\boxed{r=-1}$

$y = C_1 \cdot e^{0x} + C_2 e^{-1x}$

$= C_1 \cdot 1 + C_2 e^{-x}$

Step 2: find y_p : $y_p = x(Ax + B)$ $\boxed{\text{degree } 1}$

they give $\cdot$ (polynomial)

"make $y_p = 5x^2$: $y_p = x^2(Ax^2 + Bx + C)$

$y_p = x \cdot (Ax + B) = Ax^2 + Bx$: $y_p' = 2Ax + B$

$y_p'' = 2A$

$y'' + y' = 2x$:

$2A + (2Ax + B) = 4x$ so $2Ax = 4x \Rightarrow \boxed{A = 2}$

$\boxed{2A + B = 0}$ b/c no constant term on the right!

$B = -2A = -2 \cdot (2) = -4$

$y_p = 2x^2 - 4x$

Exercise 2. Use the method of undetermined coefficients to solve the following differential equation:

$$y'' + 6y' + 9y = 2e^{-x}$$

Answer: $y = \frac{1}{2}e^{-x} + C_1 \times e^{-3x} + C_2 \times xe^{-3x}$

Exercise 3. Use the method of undetermined coefficients to solve the following differential equation:

$$y'' + 4y = 4x$$

Answer: $y = x + C_1 \times \cos(2x) + C_2 \times \sin(2x)$

Homogeneous: $y'' + 4y = 0$: $r^2 + 4 = 0$ "imaginary"

$r = 0 \pm 2i$

$y = C_1 e^0 (\cos \beta x) + C_2 e^0 \sin \beta x$

$= C_1 \cos 2x + C_2 \sin 2x$

$\boxed{y_p = Ax}$: $y_p' = A$, $y_p'' = 0$

$0 + 4Ax = 4x \Rightarrow \underline{\underline{A = 1}}$

$\boxed{y_p = x}$

$y_p = Ax + B$: $y_p' = A$, $y_p'' = 0$

$0 + 4(Ax + B) = 4x$

$4Ax = 4x$, $4B = 0$

$A = 1$ $B = 0$

Exercise 4. Use the method of undetermined coefficients to solve the following differential equation:

$$y'' + 6y' + 9y = \boxed{2\sin(x)} = \boxed{4\cos x} \quad \text{"}y\text{"} \to y_p = A\cos x + B\sin x$$

Answer: $y = .16\sin(x) - .12\cos(x) + C_1 \times e^{-3x} + C_2 \times xe^{-3x}$

Step 1: Homogeneous: $y'' + 6y' + 9y = 0$: $r^2 + 6r + 9 = 0$: $(r+3)^2 = 0$

$\boxed{r = -3}$

$$y = C_1 e^{-3x} + C_2 \cdot x \cdot e^{-3x}$$

Step 2: Particular: $y_p = ?$

$$y_p = A\sin x + B\cos x$$

$$y_p' = A\cos x - B\sin x$$

$$y_p'' = -A\sin x - B\cos x$$

$$y'' + 6y' + 9y = 2\sin x$$

$$(-A\sin x - B\cos x) + 6(A\cos x - B\sin x) + 9(A\sin x + B\cos x) = 2\sin x$$

$$(8A - 6B)\sin x = 2\sin x$$

$$(6A + 8B)\cos x = 0$$

$$\boxed{\left.\begin{array}{l} 8A - 6B = 2 \\ 6A + 8B = 0 \end{array}\right\}}$$

$$A = .16, \quad B = -.12$$

Exercise 5. Solve the following differential equation:

trig formula

$$y'' + 4y = \boxed{\sin^3 x} = \frac{3}{4}\sin x - \frac{1}{4}\sin 3x$$

Hint: $\sin^3(x) = \frac{3\sin(x) - \sin(3x)}{4}$

Answer: $y = \frac{1}{4}\sin(x) + \frac{1}{20}\sin(3x) + C_1 \times \cos(2x) + C_2 \times \sin(2x)$

Step 1: Homogeneous: $y'' + 4y = 0$: $r^2 + 4 = 0$: $r = \pm 2i$

$r = \frac{0 \pm \sqrt{-16}}{2} = 0 \pm 2i$ $\quad r = 0 \pm 2i$ $\quad \alpha = 0$ $\quad \beta = 2$ $\quad | \; e^{\alpha}(\cos\beta x + \sin\beta x)$

$$y = C_1 e^0 \sin 2x + C_2 e^0 \cos 2x$$

$$\boxed{y = C_1 \sin 2x + C_2 \cos 2x}$$

Step 2: Find y_p :

$$y_p = A\sin x + B\sin 3x.$$

$$y_p' = A\cos x + 3B\cos 3x$$

$$y_p'' = -A\sin x - 9B\sin 3x$$

$$y'' + 4y = \frac{3}{4}\sin x - \frac{1}{4}\sin 3x$$

$$(-A\sin x - 9B\sin 3x) + 4(A\sin x + B\sin 3x) = \frac{3}{4}\sin x - \frac{1}{4}\sin 3x$$

$$3A\sin x = \frac{3}{4}\sin x \Rightarrow A = \frac{1}{4}$$

$$-5B\sin 3x = -\frac{1}{4}\sin 3x \rightarrow B = \frac{1}{20}$$

$$y_p = \frac{1}{4}\sin x + \frac{1}{20}\sin 3x$$