MATH 031 Applied Linear Algebra
Quiz #2 Detailed Solutions

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Question 1

Determine whether the vectors

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \qquad v_2 = \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \qquad v_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \end{bmatrix}$$

in $\mathbb{R}^4$ are linearly independent or linearly dependent.

If they are linearly dependent, exhibit a linear dependence relation between them.

Solution

To determine whether the vectors are linearly independent, we check whether the equation

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$$

has only the trivial solution

$$c_1 = c_2 = c_3 = 0.$$

If the only solution is the trivial solution, then the vectors are linearly independent.

If there is a nonzero solution, then the vectors are linearly dependent.

Substitute the vectors into the equation:

$$c_1 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Now distribute the constants:

$$\begin{bmatrix} c_1 \\ 0 \\ -c_1 \\ c_1 \end{bmatrix} + \begin{bmatrix} 2c_2 \\ 3c_2 \\ 0 \\ -c_2 \end{bmatrix} + \begin{bmatrix} 0 \\ c_3 \\ 0 \\ 2c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Adding the vectors gives

$$\begin{bmatrix} c_1 + 2c_2 \\ 3c_2 + c_3 \\ -c_1 \\ c_1 - c_2 + 2c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Therefore, we get the system of equations:

$$\begin{cases} c_1 + 2c_2 = 0, \\ 3c_2 + c_3 = 0, \\ -c_1 = 0, \\ c_1 - c_2 + 2c_3 = 0. \end{cases}$$

From the third equation,

$$-c_1 = 0.$$

Thus,

$$c_1 = 0.$$

Substitute $c_1 = 0$ into the first equation:

$$c_1 + 2c_2 = 0.$$

So

$$0 + 2c_2 = 0.$$

Therefore,

$$2c_2 = 0,$$

which gives

$$c_2 = 0.$$

Now substitute $c_2 = 0$ into the second equation:

$$3c_2 + c_3 = 0.$$

So

$$3(0) + c_3 = 0.$$

Thus,

$$c_3 = 0.$$

Therefore, the only solution is

$$c_1 = 0, \qquad c_2 = 0, \qquad c_3 = 0.$$

Since the only solution is the trivial solution, the vectors are linearly independent.

$$\boxed{v_1,\ v_2,\ v_3 \text{ are linearly independent.}}$$

Alternative Matrix Method

We can also place the vectors as columns of a matrix:

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \\ -1 & 0 & 0 \\ 1 & -1 & 2 \end{bmatrix}.$$

We row-reduce the matrix:

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \\ -1 & 0 & 0 \\ 1 & -1 & 2 \end{bmatrix}.$$

Use

$$R_3 \leftarrow R_3 + R_1, \qquad R_4 \leftarrow R_4 - R_1.$$

Then

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \\ 0 & 2 & 0 \\ 0 & -3 & 2 \end{bmatrix}.$$

Now use

$$R_2 \leftarrow \frac{1}{3} R_2.$$

Then

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & \frac{1}{3} \\ 0 & 2 & 0 \\ 0 & -3 & 2 \end{bmatrix}.$$

Use

$$R_1 \leftarrow R_1 - 2R_2, \qquad R_3 \leftarrow R_3 - 2R_2, \qquad R_4 \leftarrow R_4 + 3R_2.$$

Then

$$\begin{bmatrix} 1 & 0 & -\frac{2}{3} \\ 0 & 1 & \frac{1}{3} \\ 0 & 0 & -\frac{2}{3} \\ 0 & 0 & 3 \end{bmatrix}.$$

Since the third column has a pivot, we can continue reducing and obtain a pivot in every column.

This means the columns are linearly independent.

Therefore,

$$\boxed{v_1,\ v_2,\ v_3 \text{ are linearly independent.}}$$

Question 2

Let
$$T : \mathbb{R}^2 \to \mathbb{R}^3$$
be a linear transformation.

Suppose u and v are vectors in $\mathbb{R}^2$ such that
$$T(u) = (1, 3, 5)$$
and
$$T(v) = (-1, 0, 3).$$

Find the outputs:
$$\text{(a) } T(-2u), \qquad \text{(b) } T(3v), \qquad \text{(c) } T(u + 2v).$$

Important Linear Transformation Rules

Since T is linear, it satisfies two important rules.

First, scalar multiplication:
$$T(cw) = cT(w).$$

Second, addition:
$$T(w_1 + w_2) = T(w_1) + T(w_2).$$

We will use these rules to compute each output.

Part (a)

We want to compute
$$T(-2u).$$

Since T is linear,
$$T(-2u) = -2T(u).$$

We are given that
$$T(u) = (1, 3, 5).$$

Therefore,
$$T(-2u) = -2(1, 3, 5).$$

Multiply each component by -2:
$$T(-2u) = (-2, -6, -10).$$

Thus,
$$\boxed{T(-2u) = (-2, -6, -10).}$$

Part (b)

We want to compute

$$T(3v).$$

Since T is linear,

$$T(3v) = 3T(v).$$

We are given that

$$T(v) = (-1, 0, 3).$$

Therefore,

$$T(3v) = 3(-1, 0, 3).$$

Multiply each component by 3:

$$T(3v) = (-3, 0, 9).$$

Thus,

$$\boxed{T(3v) = (-3, 0, 9).}$$

Part (c)

We want to compute

$$T(u + 2v).$$

Since T is linear,

$$T(u + 2v) = T(u) + T(2v).$$

Also, by scalar multiplication,

$$T(2v) = 2T(v).$$

So

$$T(u + 2v) = T(u) + 2T(v).$$

We are given

$$T(u) = (1, 3, 5)$$

and

$$T(v) = (-1, 0, 3).$$

Therefore,

$$T(u + 2v) = (1, 3, 5) + 2(-1, 0, 3).$$

First compute

$$2(-1, 0, 3) = (-2, 0, 6).$$

Now add:

$$(1, 3, 5) + (-2, 0, 6) = (-1, 3, 11).$$

Thus,

$$\boxed{T(u + 2v) = (-1, 3, 11).}$$

Final Answers

Question 1

The equation

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$$

only has the trivial solution

$$c_1 = c_2 = c_3 = 0.$$

Therefore,

$$\boxed{v_1,\ v_2,\ v_3 \text{ are linearly independent.}}$$

Question 2

$$\boxed{T(-2u) = (-2, -6, -10)}$$

$$\boxed{T(3v) = (-3, 0, 9)}$$

$$\boxed{T(u + 2v) = (-1, 3, 11)}$$

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