Contents

Applied Methods summary
Course with Dr.Jia Gou - Spring 2022

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1 First part: Calculus of Variation

1.1 Day 1

1.1.1 Introduction

Definition 1 (functional)**.** A functional is a function defined on a domain of functions, which takes functions and gives numbers. It is usually denoted J.

Example 2. We use J to denote a functional.

1. $J : C(\mathbb{R}, \mathbb{R}) \to \mathbb{R}$, where $C(\mathbb{R}, \mathbb{R})$ is the space of continuous real value functions between $\mathbb{R}$ and $\mathbb{R}$.
2. $J : C(\mathbb{R}^m, \mathbb{R}^n) \to \mathbb{R}$
3. $J[y] = y(3)$. If $y_1 = x^2$ and $y_2 = e^x$, then:
$$\begin{cases} J[y_1] = 3^2 = 9 & \text{if } y_1 = x^2 \\ J[y_2] = e^3 & \text{if } y_2 = e^x \end{cases}$$
4. Let $y(x)$ be an arbitrary continuous differentiable function defined on the interval $[a, b]$. We define:
$$J[y] = \int_a^b (y'(x))^2 dx$$
Then $J[y(x)]$ is a functional defined on the set of all such functions.

Example 3 (Arc Length)**.** Let $J[y(x)]$ be the length of a curve in $\mathbb{R}^2$ connecting $A(x_0, y(x_0))$ and $B(x_1, y(x_1))$. How do we get a formula for $J[y]$? And is there a minimum shortest length connecting two points?
By Pythagorean (see figure (1) for reference):
$$ds^2 = dx^2 + dy^2 = dx^2 + (y'(x)dx)^2$$
Therefore
$$ds = \sqrt{dx^2 + (y'dx)^2} = \sqrt{(1 + (y')^2)dx^2} = \sqrt{1 + (y')^2}dx$$
Hence we get:
$$J[y(x)] = \int ds = \int_{x_0}^{x_1} \sqrt{1 + (y')^2}dx$$

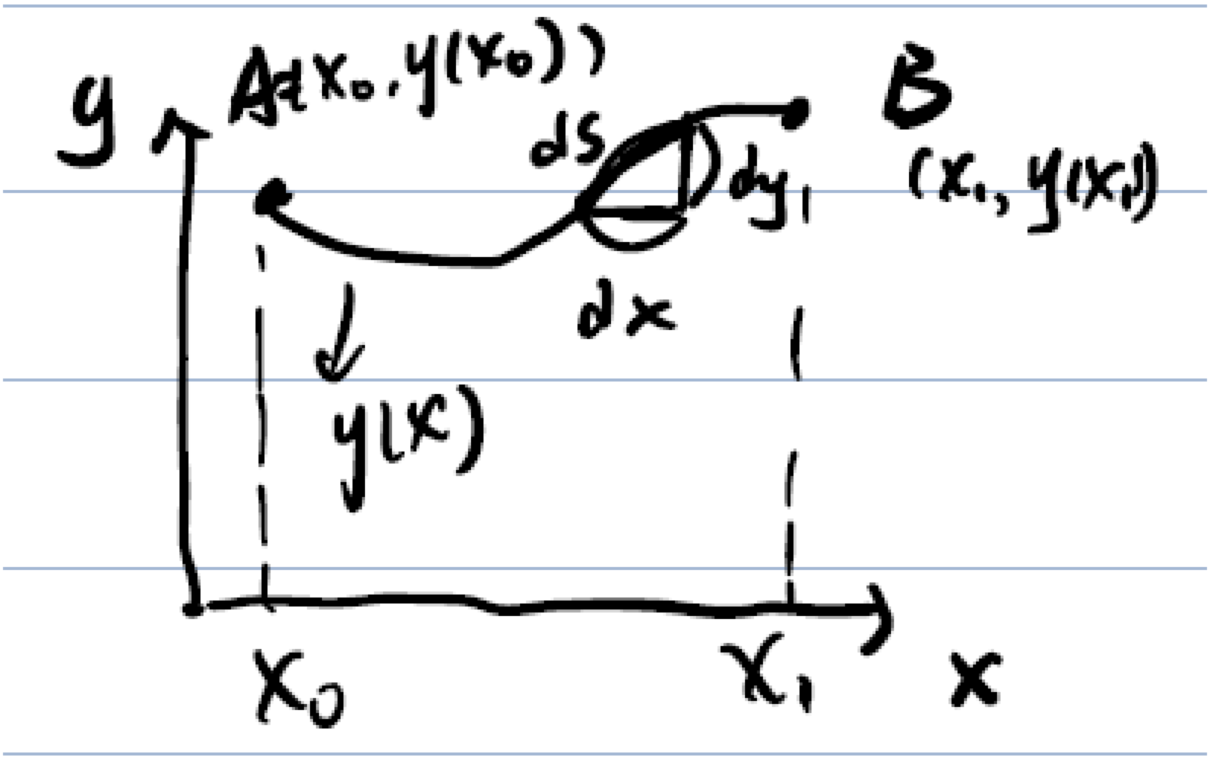

Figure 1: Arc length

Example 4 (**Brachistochrone Problem**). Consider a frictionless small ball sliding down a wire connecting A and B and moving only through the force of gravity. Suppose that there is no initial velocity. The functional T that describe the travel time can be calculated as:

$$\text{time} = \frac{\text{distance}}{\text{velocity}}$$

In math symbol:

$$T = \int_C \frac{ds}{v}$$

Similar to earlier example, $ds = \sqrt{1 + (y')^2}dx$. It remains to find dv. This requires some physics (see figure (2) for reference). The Kinetic Energy at B minus the Kinetic Energy at A is the total work done in moving the ball from A to B. So:

$$\begin{aligned}
\frac{1}{2}mv^2 - 0 &= mgy \\
\frac{1}{2}mv^2 &= mgy \\
v^2 &= 2gy \\
v &= \sqrt{2gy}
\end{aligned}$$

Now put in T and we get the equation:

$$T[y] = \int_{x_0}^{x_1} \frac{\sqrt{1 + (y')^2}dx}{\sqrt{2gy}}$$

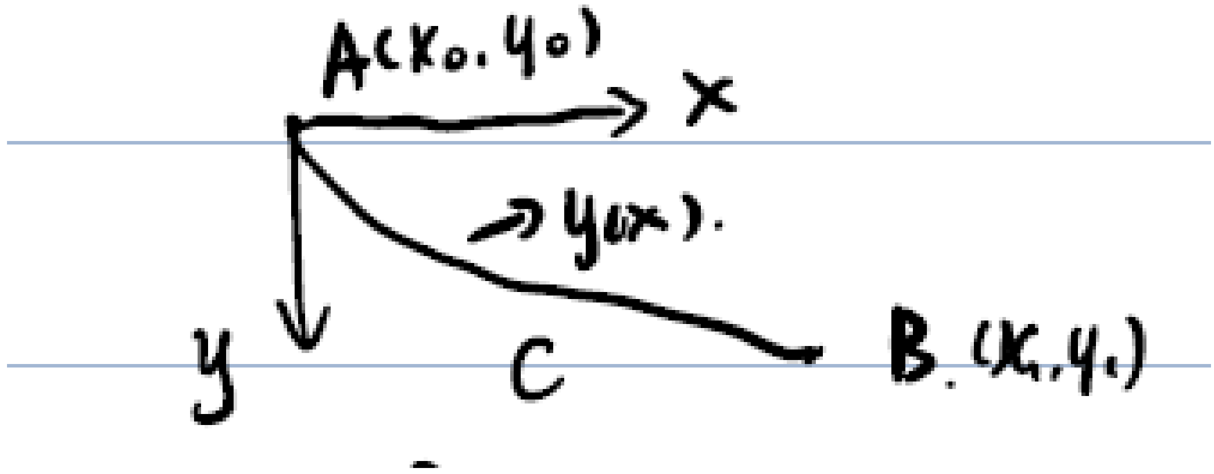

Figure 2: Brachistochrone Problem

Question 5. In Brachistochrone problem, can we:

1. Find $y(x)$ so that it gives the shortest travel time?
2. What if the case the initial velocity of the ball at A is non-zero, what is T formula?

An important class of functional, the **linear functional**:

Definition 6 (linear functional)**.** Let $J[f]$ be a functional defined on a normed linear space K. Then $J[f]$ is said to be a **linear functional** if:

1. $J[\alpha f] = \alpha J[f]$, for any $f \in K$ and any real number α.
2. $J[f_1 + f_2] = J[f_1] + J[f_2]$, for any $f_1, f_2 \in K$.

1.1.2 Foundational

Next, we will begin the theory of variation. A building foundation (which helps give the proof of Euler-Lagrange Equation) is the Fundamental Lemma of Calculus of Variation.

Lemma 7 (**Fundamental Lemma of Calculus of Variation**)**.** Let $g(x) \in C[a, b]$ satisfying the condition:

$$\int_a^b g(x)\eta(x)dx = 0$$

for all $\eta(x) \in C[a, b]$ such that $\eta(a) = \eta(b) = 0$. Then we must have:

$$g(x) = 0 \text{ for all } x \in [a, b].$$

Proof. The proof idea is by contradiction. Suppose there is a point x^* such that $g(x^*) \neq 0$. WLOG, assume $g(x^*) > 0$. As $g(x)$ is continuous on $[a, b]$, there must exists a small interval $x^* \in [x_1, x_2] \subseteq [a, b]$ such that $g(x) > 0$ for all $x \in [x_1, x_2]$. Now, if we set our η function to be:

$$\eta(x) = \begin{cases} (x - x_1)(x - x_2) & \text{if } x \in [x_1, x_2] \\ 0 \text{ otherwise} \end{cases}$$

Then $\eta(x)$ is continuous on (a,b) and also $\eta(a)=\eta(b)=0$. Then:

$$\begin{aligned}\int_a^b g(x)\eta(x)dx &= \int_{x_0}^{x_1} g(x)\eta(x)dx \\ &= \int_{x_0}^{x_1} g(x)(x-x_1)(x-x_2) > 0 \\ &\text{since the integrand is positive except at end points } x_1 \text{ and } x_2\end{aligned}$$

And this is a contradiction. Hence we proved the Lemma. □

Lemma 8 (Second Fundamental Lemma of Calculus of Variation)**.** Let $g(x)$ be continuous in $[a,b]$ and that if $\int_a^b g(x)\eta'(x)dx = 0$ for every $\eta(x) \in C^1(a,b)$ such that $\eta(a)=\eta(b)=0$. Then

$$g(x) = C \text{ for all } x \in [a,b]$$

Hints. Let $\eta(x) = \int_a^x (g(\xi)-c)d\xi$, and C is defined as $\int_a^b (g(x)-c) = 0$. Then consider:

$$\int_a^b (g(x)-C)\eta'(x)dx$$

Proof. HW. □

Definition 9 (Extremum)**.** If $J[y]$ is a functional. Suppose there exists some $y_0(x)$ such that $J[y]-J[y_0]$ does **not** change its sign for any $y(x)=y_0(x)+\varepsilon\eta(x)$ for sufficiently small ε. Then we say $J[y]$ has an **extremum** for $y=y_0$.(see figure (3) for reference)

Question 10. How can we find the location of the extremum of a functional $J[y]$? We use the **variation** of J technique.

Definition 11 (variation of J and extremal of J)**.** Consider a family of functions $y_\varepsilon = y_0 + \varepsilon\eta(x)$ for some specific y_0 and η. Let

$$J[y(x)] := \int_a^b F(x,y,y')dx$$

and

$$I(\varepsilon) = J[y_o + \varepsilon\eta] = \int_a^b F(x, y_0+\varepsilon\eta, y_0'+\varepsilon\eta')dx$$

Then the **variation** of J at y_0 is defined to be:

$$\delta J[y_0] = \left.\frac{dI}{d\varepsilon}\right|_{\varepsilon=0}$$

Observe that since I is just a function of the single variable ε, therefore when $\varepsilon=0$ in the above definition, we will get an **extreme value**. So we say y_0 is an **extremal** of J if:

$$\frac{d}{d\varepsilon}J[y_0+\varepsilon\eta] = \left.\frac{dI}{d\varepsilon}\right|_{\varepsilon=0} = 0$$

Remark 12. The extremal condition above is a **necessary condition**: If a function $f(x)$ has a min/max at x_0, then $f'(x_0)=0$ if the derivative exists.

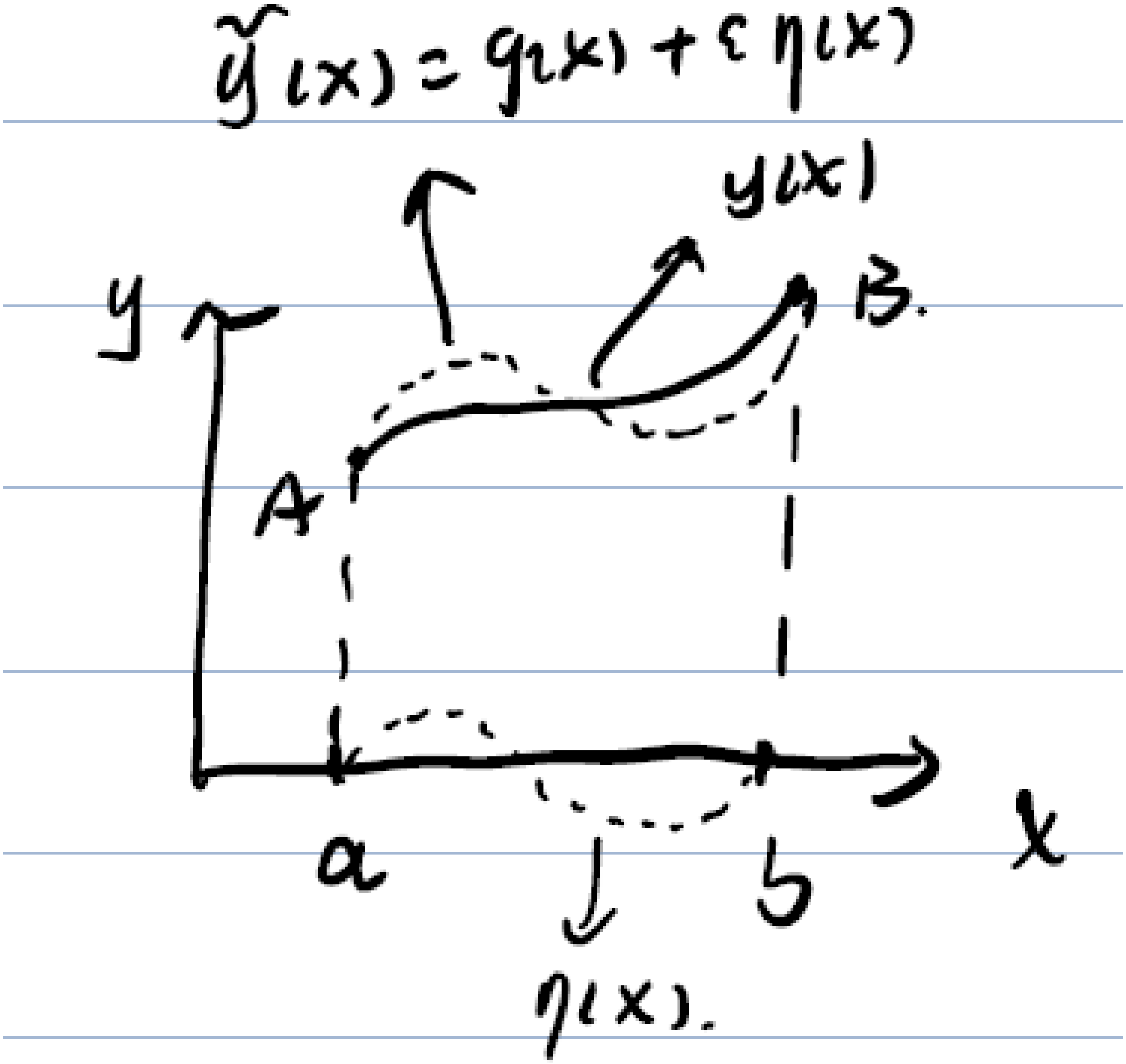

Figure 3: Variation of J

1.2 Day 2

1.2.1 Euler-Lagrange Equation

Theorem 13 (The Euler-Lagrange Equation or Euler's equation)**.** *Consider the functional:*

$$J([y(x)] = \int_a^b F(x, y, y')dx$$

defined on all functions $y(x)$ *with* $y(a) = A$ *and* $y(b) = B$*. If* $J[y]$ *has an extremum at* $y(x)$*, then* $y(x)$ *satisfies the E-L equations (Euler equation):*

$$F_y - \frac{d}{dx}F_{y'} = 0 \tag{1}$$

Equivalently, the computation is:

$$F_y - F_{y'x} - F_{y'y}y' - F_{y'y}y'' = 0$$

Remark 14. Remember the formula. And this is for the case of 2 fixed end points.

Proof. This might be skipped while studying for qual?. The $y(x)$ gives an extremum of J. We defined:

$$\tilde{y}(x) = y(x) + \varepsilon\eta(x)$$

We know that $y(a) = A$ and $y(b) = B$, we require: $\tilde{y}(a) = A$ and $\tilde{y}(b) = B$.This gives $\eta(a) = \eta(b) = 0$. Now since $y(x)$ gives an extremum of J, we must get $\delta J[y] = \frac{d}{d\varepsilon}J[\tilde{y}]\Big|_{\varepsilon=0} = 0$:

$$\begin{aligned}
\frac{d}{d\varepsilon}J[\tilde{y}] &= \frac{d}{d\varepsilon}\int_a^b F(x, \tilde{y}, \tilde{y}')dx \\
&= \int_a^b \frac{d}{d\varepsilon}F(x, \tilde{y}, \tilde{y}')dx \\
&= \int_a^b \left(\frac{\partial F}{\partial x}\frac{dx}{d\varepsilon} + \frac{\partial F}{\partial \tilde{y}}\frac{\partial \tilde{y}}{d\varepsilon} + \frac{\partial F}{\partial \tilde{y}'}\frac{\partial \tilde{y}'}{d\varepsilon}\right)dx \\
&= \int_a^b \left(\frac{\partial F}{\partial \tilde{y}}.\eta(x) + \frac{\partial F}{\partial \tilde{y}'}.\eta'(x)\right)dx
\end{aligned}$$

The last equality is true because $\frac{dx}{d\varepsilon} = 0$, and $\frac{\partial \tilde{y}}{d\varepsilon} = \eta(x)$, and $\frac{\partial \tilde{y}'}{d\varepsilon} = \eta'$. Now we calculate the second term above by integration by part:

$$\begin{aligned}
\int_a^b \frac{\partial F}{\partial \tilde{y}'}.\eta'(x)dx &= \frac{\partial F}{\partial \tilde{y}'}.\eta(x)\Big|_a^b - \int_a^b \frac{d}{dx}(\frac{\partial F}{\partial \tilde{y}'})\eta(x)dx \\
&= -\int_a^b \frac{d}{dx}(\frac{\partial F}{\partial \tilde{y}'})\eta(x)dx \\
&\text{since } \eta(a) = \eta(b) = 0
\end{aligned}$$

So

$$\begin{aligned}
\frac{d}{d\varepsilon}J[\tilde{y}]\bigg|_{\varepsilon=0} &= \int_a^b \frac{\partial F}{\partial \tilde{y}}.\eta(x) - \frac{d}{dx}(\frac{\partial F}{\partial \tilde{y}'})\eta(x)dx\bigg|_{\varepsilon=0} \\
&= \int_a^b \left(\frac{\partial F}{\partial \tilde{y}} - \frac{d}{dx}(\frac{\partial F}{\partial \tilde{y}'})\right)\eta(x)dx\bigg|_{\varepsilon=0} \\
&= \int_a^b \left(\frac{\partial F}{\partial y} - \frac{d}{dx}(\frac{\partial F}{\partial y'})\right)\eta(x)dx = 0 \\
&\text{when } \varepsilon = 0, \tilde{y} = y.
\end{aligned}$$

Now if F is sufficiently smooth, $\frac{\partial F}{\partial \tilde{y}} - \frac{d}{dx}(\frac{\partial F}{\partial \tilde{y}'})$ is continuous. Since $\eta(x)$ is arbitrary, by the Fundamental Theorem of Calculus of Variation, we must have the function which we are multiplying for $\eta(x)$ must be zero:

$$\frac{\partial F}{\partial y} - \frac{d}{dx}(\frac{\partial F}{\partial y'}) = 0$$

And this is the E-L equation (with $y(a) = A$ and $y(b) = B$. □

Remark 15. A smooth extremal might not exist.

Definition 16. A function $y(x)$ is called an extremal of a functional $J[y] = \int_a^b F(x, y, y')dx$ if it satisfies the E-L equation.

Remark 17 (Variational derivative of a functional)**.** The $F_y - \frac{d}{dx}F_{y'} = 0$ is called the **variational derivative of the functional** J.

Remark 18 (**Beltrami identity**)**.** A special case of E-L equation is when F does **not** depends on x, says $F = F(y, y')$ and this is called the Beltrami identity:

$$F - y'F_{y'} = C \tag{2}$$

To see why this is true: We begin with the usual E-L equation:

$$\begin{aligned}
F_y - \frac{d}{dx}F_{y'} &= 0 \\
&\text{Multiply by } y' \text{ gives} \\
y'F_y - y'\frac{d}{dx}F_{y'} &= 0
\end{aligned}$$

Now note that $F_x = 0$ (since F does not depends on x), so in the above equation we add the term: $F_x + F_{y'}y'' - F_{y'}y''$ and then regroup:

$$\begin{aligned}
&F_x + y'F_y + F_{y'}y'' - F_{y'}y'' - y'\frac{d}{dx}F_{y'} = 0 \\
\iff &\frac{d}{dx}(F - y'F_{y'}) = 0 \\
\iff &F - y'F_{y'} = C
\end{aligned}$$

Question 19. How about when F does not depend on y or y'?

Example 20 (Work-out Shortest length)**.** Let $J[y(x)]$ be the length of a curve in $\mathbb{R}^2$ connecting $A(x_0, y(x_0))$ and $B(x_1, y(x_1))$. Find the $y(x)$ that gives the shortest length. Recall that

$$J[y(x)] = \int_{x_0}^{x_1} \sqrt{1 + (y')^2} dx$$

Then we let

$$F(x, y, y') = \sqrt{1 + (y')^2} = F(y')$$

Let's compute some term to put in E-L equation:

1. F_y=0
2. The term $F_{y'}$ is:

$$\begin{aligned} F_{y'} &= \frac{d}{dy'}(1 + (y')^2)^{\frac{1}{2}} = \frac{1}{2}(1 + (y')^2)^{-\frac{1}{2}} 2y' \\ &= \frac{y'}{\sqrt{1 + (y')^2}} \end{aligned}$$

Now we put into E-L equation:

$$F_y - \frac{d}{dx} F_{y'} = 0$$

gives:

$$0 - \frac{d}{dx} \frac{y'}{\sqrt{1 + (y')^2}} = 0$$

Therefore:

$$\frac{y'}{\sqrt{1 + (y')^2}} = 0$$

And so:

$$\frac{y'}{\sqrt{1 + (y')^2}} = C$$

We will now solve this by multiplying and squaring:

$$\begin{aligned} (y')^2 &= C^2(1 + (y')^2) \\ (y')^2(1 - C^2) &= C^2 \\ y' &= \sqrt{\frac{C^2}{1 - C^2}} = m \end{aligned}$$

Hence we must have $y = mx + b$. We can determine the value of m and b based on the initial conditions $A(x_0, y_0)$ and $B(x_1, y_1)$. This gives:

$$y = \frac{y_0 - y_1}{x_0 - x_1} x + (y_1 - \frac{y_0 - y_1}{x_0 - x_1} x_1)$$

Example 21 (Brachistochrone Problem - full worked out)**.** Consider a frictionless small ball sliding down a wire connecting A and B and moving only through the force of gravity. Suppose that there is no initial velocity. What is the shape of the wire so that the ball has the shortest travel time?
WLOG, we may assume $A(0,0)$ and $B(x_1, y_1)$. Again recall earlier, the travel time is a functional T which we derived:

$$\begin{aligned} T[y(x)] &= \int_0^{x_1} \frac{\sqrt{1+(y')^2}dx}{\sqrt{2gy}} \\ &= \frac{1}{\sqrt{2g}} \int_0^{x_1} \frac{\sqrt{1+(y')^2}dx}{\sqrt{y}} \end{aligned}$$

So if we denote

$$F = \frac{\sqrt{1+(y')^2}}{\sqrt{y}}$$

We start the computation in E-L equation, note that there is no x in it, so we'll use the Beltrami Identiy:

$$F - y'F_{y'} = C$$

We compute:

$$F_{y'} = \frac{y'}{\sqrt{1+(y')^2}} . \frac{1}{\sqrt{y}}$$

Then put it into Beltrami equation:

$$\begin{aligned} \frac{\sqrt{1+(y')^2}}{\sqrt{y}} - y' . \frac{y'}{\sqrt{y}\sqrt{1+(y')^2}} &= C \\ \frac{1+(y')^2-(y')^2}{\sqrt{y}\sqrt{1+(y')^2}} &= C \\ y(1+(y')^2) &= \frac{1}{C^2} = k \end{aligned}$$

Now we try to solve this ODE problem, using separable method:

$$\begin{aligned} y(1+(y')^2) &= k \\ 1+(y')^2 &= \frac{k}{y} \\ \frac{dy}{dx} = y' &= \sqrt{\frac{k-y}{y}} \\ \int dx &= \int \sqrt{\frac{k-y}{y}} dy \end{aligned}$$

Now from calculus, the integration on the right is using the trig substitution $y = k \sin^2 \theta$

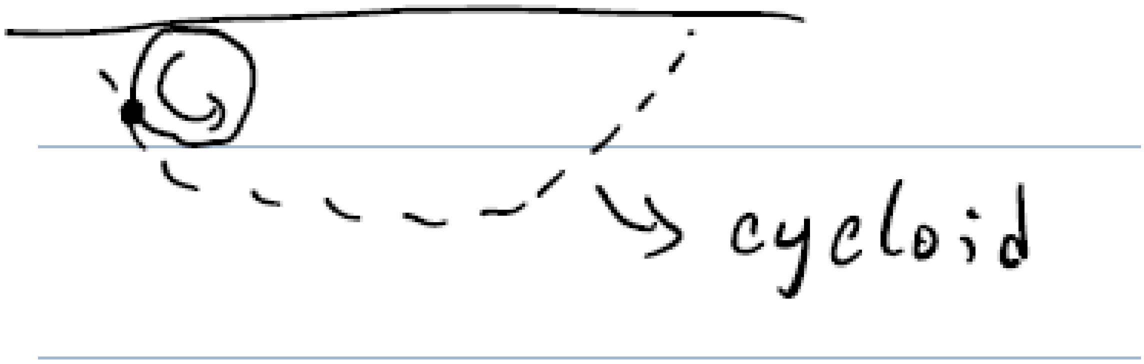

Figure 4: Cycloid

and $dy = k2\sin\theta\cos\theta$:

$$\begin{aligned} x &= \int \sqrt{\frac{k\sin^2\theta}{k\cos^2\theta}} 2k\sin\theta\cos\theta d\theta \\ &= k\int 2\sin^2\theta d\theta \\ &= k\int 1-\cos(2\theta)d\theta = k(\theta - \frac{\sin 2\theta}{2}) + C \end{aligned}$$

Therefore we get:

$$y = k\sin^2\theta = k\frac{1}{2}(1-\cos 2\theta)$$

Since we have 1 initial point $(0,0)$ for (x,y), we get that $C=0$. Se we have the following curve, which is a cycloid, see figure (4) for picture (in parametric form):

$$\begin{cases} y &= k\frac{1}{2}(1-\cos 2\theta) \\ x &= k(\theta - \frac{\sin 2\theta}{2}) \end{cases}$$

Remark 22. What happens if the initial velocity of the ball is non-zero?

1.2.2 Special class of Euler-Lagrange: no boundary conditions

Remark 23. In the previous section, we derived the E-L equation in the case that $y(a) = A$ and $y(b) = B$ to get $\eta(a) = \eta(b) = 0$. That means we have 2 **fixed boundary points**. What happens if we have **no boundary condition**?

We establish the theory, starting with the following example:

Example 24 (Euler-Lagrange for variable end points)**.** Suppose end points of $y(x)$ lie on two vertical lines $x = a$ and $x = b$. Find the curve for which the functional $J[y]$ has an extremum. We follow the proof of Euler- Lagrange general case and consider this:

$$\delta I = \frac{d}{d\varepsilon}J[y+\varepsilon\eta]\Big|_{\varepsilon=0} = \int_a^b (F_y - \frac{d}{dx}F_{y'})\eta(x)dx + F_{y'}\Big|_{x=b}\eta(b) - F_{y'}\Big|_{x=a}\eta(a) = 0$$

For a special class of $\eta(x)$: $\eta(a) = \eta(b) = 0$, the variation $\delta I = 0$. Then in order for the term δI to be zero, we must have the following:

1. The first integral is zero:

$$\int_a^b (F_y - \frac{d}{dx}F_{y'})\eta(x)dx = 0$$

Therefore the integrand must also be zero. And since $\eta(x)$ can be any function, we must have:

$$F_y - \frac{d}{dx}F_{y'} = 0$$

And this means that $y(x)$ **satisfies the Euler-Lagrange equation**. And this gives $y(x)$ must be an extremal.

2. The second term must also be zero, that is:

$$F_{y'}\Big|_{x=b}\eta(b) - F_{y'}\Big|_{x=a}\eta(a)$$

As $\eta(x)$ is arbitrary, we must have:

$$F_{y'}\Big|_{x=b} = 0$$

and

$$F_{y'}\Big|_{x=a} = 0$$

This is called the **natural boundary condition**.

1.2.3 Multi-dimensional functional

A. Consider the functional (one variable x and n functions of x the $y_i(x)$):

$$J[y_1, \ldots, y_n] = \int_a^b F(x, y_1, \ldots, y_n, y_1', \ldots, y_n')dx$$

A neccessary condition for the curve $y_i = y_i(x)$, for $i = 1, 2, \ldots, n$ to be an extremal of the functional $J[y_1, y_2, \ldots, y_n]$ is that the functions $y_i(x)$ satisfy the Euler equations:

$$F_{y_i} - \frac{d}{dx}F_{y_i'} = 0, \text{ for } i = 0, 1, 2, \ldots, n$$

Example 25. Consider

$$F = y_1y_2^2 + y_1^2y_2 + y_1'y_2'$$

And $J[y_1, \ldots, y_n] = \int_a^b F dx$. The E-L equations are:

1. $F_{y_1} - \frac{d}{dx}F_{y_1'} = 0$ which is equivalent to:

$$y_2^2 + 2y_1y_2 - \frac{d}{dx}y_2' = 0$$

2. $F_{y_2} - \frac{d}{dx}F_{y_2'} = 0$ which is equivalent to:

$$y_1^2 + 2y_1y_2 - \frac{d}{dx}y_1' = 0$$

B. Consider a functional of the form (2 or more independent variables, say x, y, and one function of 2 variable, say $z(x, y)$):

$$J[z] = \int\int_R F(x, y, z, z_x, z_y)dxdy \text{ with } z = z(x, y)$$

where R is some closed region. The E-L equation for the functional $J[z]$ is:

$$F_z - \frac{d}{dx}F_{z_x} - \frac{d}{dy}F_{z_y} = 0$$

Example 26.

$$F = F(x, t, y, y_x, y_t) = \frac{1}{2}y_x^2 - \frac{1}{2c^2}y_t^2$$

Then the E-L equation is:

$$\begin{aligned} F_y - \frac{d}{dx}F_{y_x} - \frac{d}{dt}F_{y_t} &= 0 \\ 0 - \frac{d}{dx}y_x - \frac{d}{dt}(-\frac{1}{2c^2}2y_t) &= 0 \\ -\frac{d^2y}{dx^2} + \frac{1}{c^2}\frac{d^2y}{dt^2} &= 0 \\ \frac{d^2y}{dt^2} &= c^2\frac{d^2y}{dx^2} \text{ which is the Wave equation} \end{aligned}$$

1.3 Day 3

1.3.1 Invariance of E-L

Next we establish that Euler-Lagrange should be the same when we do change of variables. We have the following theorem:

Theorem 27 (Invariance of the Euler-Lagrange Equations)**.** *Consider the functional:*

$$J[y(x)] = \int_a^b F(x, y, y')dx$$

with the corresponding Euler's equation:

$$F_y - \frac{d}{dx}F_{y'} = 0$$

Let $x = x(u, v)$ and $y = y(u, v)$ be new coordinates such that

$$\begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} \neq 0$$

Then if a curve $\tilde{y} = \tilde{y}(x)$ satisfies the Euler's equation, the corresponding curve $\tilde{v} = \tilde{v}(u)$ obtained from using the change of variables satisfies the Euler's equation:

$$\frac{\partial F_1}{\partial \tilde{v}} - \frac{\partial}{\partial u}\frac{\partial F_1}{\partial \tilde{v}'} = 0$$

where the F_1 defined the functional obtained from $J[y]$ using the change of variables:

$$J[y] \to J_1[y] = \int_{\alpha_1}^{\beta_1} F_1(u, v, v')du$$

Proof. Skip. The idea is that

$$J_1[y] = \int \alpha_1{}^{\beta_1} F(x(u,v), y(u,v), \frac{y_u + y_v v'}{x_u + x_v v'})(x_u + x_v v')du = \int_{\alpha_1}^{\beta_1} F_1(u,v,v')du$$

And we can continue.... $\square$

Example 28. Find the extremals of the functional $J[r] = \int_{\varphi_1}^{\varphi_2} \sqrt{r^2 + (r')^2} d\varphi$, where $r = r(\varphi)$.
We have 2 methods of solving these:

1. Method 1: Use the corresponding Euler Lagrange equation is $F_r - \frac{d}{d\varphi} F_{r'} = 0$ with $F = \sqrt{r^2 + (r')^2}$. Therefore the Euler Lagrange Equation is:

$$\frac{r}{\sqrt{r^2 + (r')^2}} - \frac{d}{d\varphi} \frac{r'}{\sqrt{r^2 + (r')^2}} = 0$$

2. Method 2: The second method is using change of variable. Let $x = r\cos\varphi$ and $y = r\sin\varphi$. Then:

$$dx = dr\cos\varphi - r\sin\varphi d\varphi$$

and

$$dy = dr\sin\varphi + r\cos\varphi d\varphi$$

and

$$dr = r' d\varphi$$

Let us calculate:

$$(dx)^2 = (dr)^2\cos^2\varphi - 2rdr\sin\varphi\cos\varphi d\varphi + r^2\sin^2\varphi (d\varphi)^2$$

and

$$(dy)^2 = (dr)^2\cos^2\varphi + 2rdr\sin\varphi\cos\varphi d\varphi + r^2\sin^2\varphi (d\varphi)^2$$

Now add them together, the term $2rdr\sin\varphi\cos\varphi d\varphi$ is cancelled and we get:

$$(dx)^2 + (dy)^2 = (dr)^2\cos^2\varphi + r^2\sin^2\varphi (d\varphi)^2$$

Now note that $dy = y'dx$, and $dr = r'd\varphi$ so the above is:

$$(1 + (y')^2)(dx)^2 = ((r')^2 + r^2)(d\varphi)^2$$
$$\sqrt{1 + (y')^2} dx = \sqrt{(r')^2 + r^2} d\varphi$$

Then the $J_1[y] = \int_{x_1}^{x_2} \sqrt{1 + (y')^2} dx$, and the solution to this E-L equation of $J_1[y]$ is $y = mx + b$. Convert it back to r we get

$$r\sin\varphi = mmr\cos\varphi + b$$

and so

$$r = \frac{b}{\sin\varphi - m\cos\varphi}$$

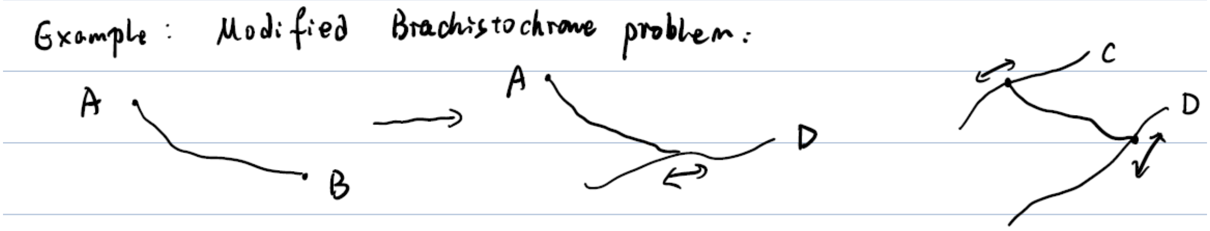

Figure 5: Modified Brachistochrone Problem

1.3.2 Euler-Lagrange when 2 end points can varies among a curve

For instance, in a modified Brachistochrone problem such as in Figure(5), we varies from having 2 fixed end points A and B to the case of 1 fixed end point A and another end point D varying on a curve, and the the last case when both end points C and D varying on 2 curves.

Now we have this theory:

Theorem 29. *Consider $J = \int_{x_1}^{x_2} F(x, y, y')dx$ where (x_1, y_1) and (x_2, y_2) can vary along two curves Y_1 and Y_2, respectively, and that:*

$$y(x_i) = Y_i(x_i) \text{ for } i = 1, 2$$

Suppose the functional J has an extremum for $y = y(x)$ with $x_{01} \leq x \leq x_{02}$, then :

1. *$y(x)$ must be an extremal of J*
2. *$y(x)$ must satisfy the* ***transversality condition****:*

$$\left[F - F_{y'}y' + F_{y'}Y_i' \right] \Big|_1^2 = 0$$

Proof. Let $\tilde{y}(x, \varepsilon) = y(x) + \varepsilon\eta(x)$ with $x_1(\varepsilon) \leq x \leq x_2(\varepsilon)$, and $-\delta \leq \varepsilon \leq \delta$. Note that $\eta(x)$, $x_1(\varepsilon), x_2(\varepsilon$ are arbitrary functions.
If we require end points of $\tilde{y}(x, \varepsilon)$ on $Y_i(x)$, then:

$$y(x_i(\varepsilon)) + \varepsilon\eta(x_i(\varepsilon)) = Y_i(x_i(\varepsilon))$$

Differentiating both sides wrt ε at $\varepsilon = 0$:

$$y'(x_i(0))\frac{dx_i(0)}{d\varepsilon} + \eta(x_i(0)) = Y_i'(x_i(0))\frac{dx_i(0)}{d\varepsilon}$$

Denoting $x_{0i} = x_i(0)$, the above is:

$$\eta(x_{0i}) = Y_i'(x_i(0))\frac{dx_i(0)}{d\varepsilon} - y'(x_{0i})\frac{dx_i(0)}{d\varepsilon}$$

With the functional

$$I(\varepsilon) = \int_{x_1(\varepsilon)}^{x_2(\varepsilon)} F(x, y + \varepsilon\eta, y' + \varepsilon\eta')dx$$

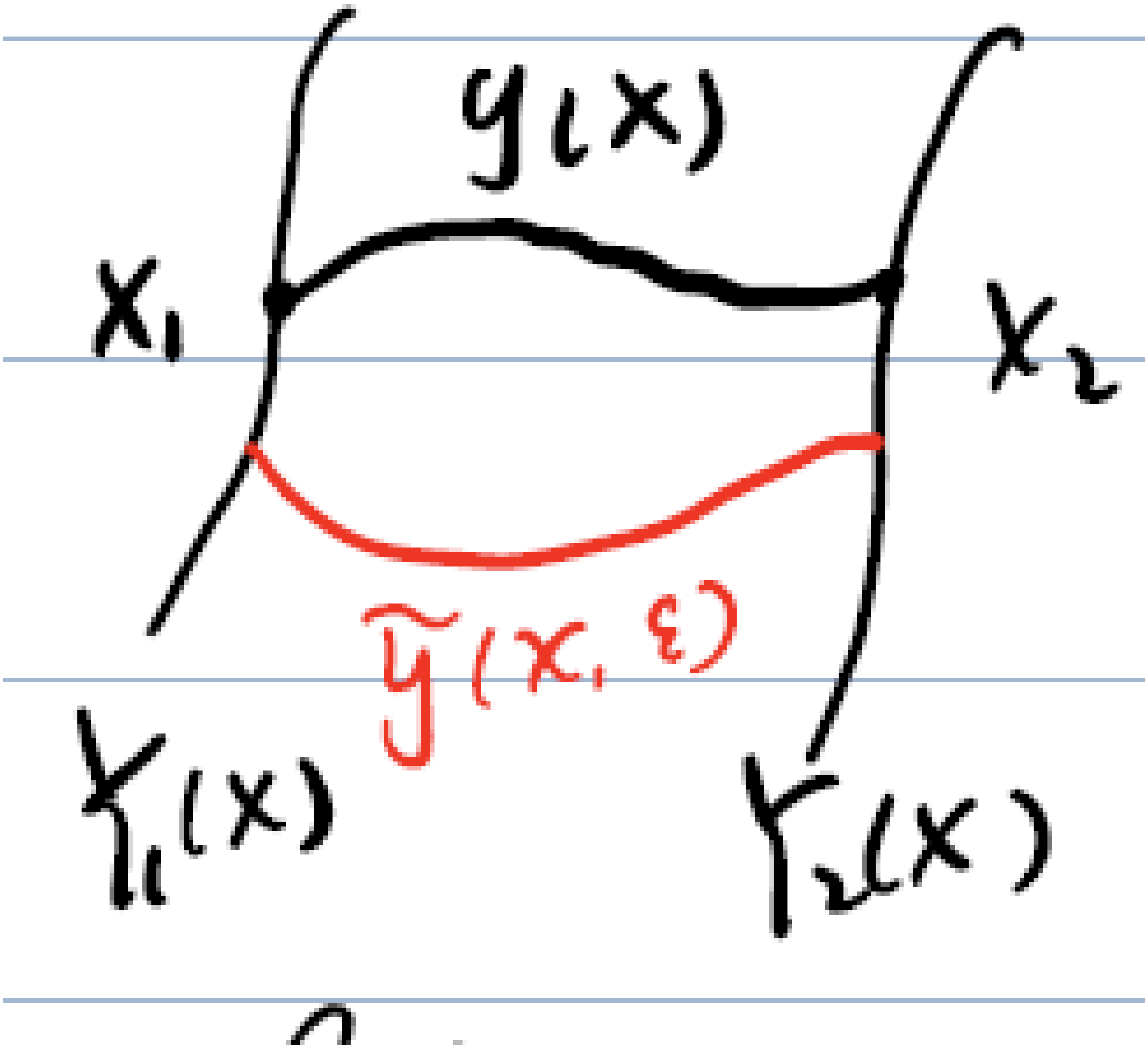

Figure 6: varying end points

To obtain the extremum value at $y(x)$, we must require $\left.\frac{dI(\varepsilon)}{d\varepsilon}\right|_{\varepsilon=0} = 0$. Then:

$$\begin{aligned}
0 &= \left.\frac{dI(\varepsilon)}{d\varepsilon}\right|_{\varepsilon=0} \\
&= \left.\left(F(x_2(\varepsilon), y+\varepsilon\eta|_{x_2(\varepsilon)}, y'+\varepsilon\eta'|_{x_2(\varepsilon)})\right)\frac{dx_2(\varepsilon)}{d\varepsilon}\right|_{\varepsilon=0} \\
&- \left.\left(F(x_1(\varepsilon), y+\varepsilon\eta|_{x_1(\varepsilon)}, y'+\varepsilon\eta'|_{x_1(\varepsilon)})\right)\frac{dx_1(\varepsilon)}{d\varepsilon}\right|_{\varepsilon=0} \\
&+ \left.\int_{x_1(\varepsilon)}^{x_2(\varepsilon)} \frac{d}{d\varepsilon}F(x, y+\varepsilon\eta, y'+\varepsilon\eta')dx\right|_{\varepsilon=0}
\end{aligned}$$

For simplicity, we write $F(x_{0i}) = F(x_{0i}, y(x_{0i}), y'(x_{0i}))$. So the above is simplified into:

$$\begin{aligned}
0 &= \left.\frac{dI(\varepsilon)}{d\varepsilon}\right|_{\varepsilon=0} \\
&= F(x_{0i})\left.\frac{dx_i(0)}{d\varepsilon}\right|_1^2 + \int_{x_1(0)}^{x_2(0)} \frac{\partial F}{\partial x}\frac{\partial x}{\partial \varepsilon} + \frac{\partial F}{\partial y}\eta + \frac{\partial F}{\partial y'}\eta' dx
\end{aligned}$$

Now note that $\frac{\partial x}{\partial \varepsilon} = 0$ and then alo use integration by part for the left over term with the integral, we get:

$$\begin{aligned}
0 &= \left.\frac{dI(\varepsilon)}{d\varepsilon}\right|_{\varepsilon=0} \\
&= F(x_{0i})\left.\frac{dx_i(0)}{d\varepsilon}\right|_1^2 + \left.\frac{\partial F}{\partial y'}\eta\right|_1^2 + \int_{x_1(0)}^{x_2(0)} \frac{\partial F}{\partial y}\eta - \frac{d}{dx}\frac{\partial F}{\partial y'}\eta dx \\
&= F(x_{0i})\left.\frac{dx_i(0)}{d\varepsilon}\right|_1^2 + \left.\frac{\partial F}{\partial y'}\left(Y_i'(x_i(0))\frac{dx_i(0)}{d\varepsilon} - y'(x_{0i})\frac{dx_i(0)}{d\varepsilon}\right)\right|_1^2 + \int_{x_1(0)}^{x_2(0)} \left(\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'}\right)\eta dx \\
&= \left.\left(F(x_{0i}) - F_{y'}(x_{0i})y'(x_{0i}))\frac{dx_i(0)}{d\varepsilon} + F_{y'}(x_{0i})Y_i'(x_i(0))\frac{dx_i(0)}{d\varepsilon}\right)\right|_1^2 + \int_{x_1(0)}^{x_2(0)} \left(\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'}\right)\eta dx
\end{aligned}$$

Choose a particular class of $\eta(x)$ such that $\eta(x_{0i}) = 0$ and $\frac{dx_i(0)}{d\varepsilon} = 0$, so we got

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} = 0 \text{ which is E-L equation}$$

And in addition the term:

$$\left.\left(F(x_{0i}) - F_{y'}(x_{0i})y'(x_{0i}))\frac{dx_i(0)}{d\varepsilon} + F_{y'}(x_{0i})Y_i'(x_i(0))\frac{dx_i(0)}{d\varepsilon}\right)\right|_1^2 = 0$$

Therefore we must have when $i = 1$ and $i = 2$, each of those two terms equal to zero:

$$\Big((F - F_{y'}y') + F_{y'}Y_i'\Big)\Big|_1^2 = 0$$

Or in term of derivatives, equivalently:

$$\Big((F - F_{y'}y')dx_i + F_{y'}dY_i'\Big)\Big|_1^2 = 0$$

And this completed the proof. $\square$

However, in practice, we may see nice class of functional, in which we can convert **transversality condition** into **orthogonality condition**.

Example 30. Consider a functional of arc length:

$$J[y] = \int_{x_0}^{x_1} \sqrt{1 + (y')^2}dx = \int_{x_0}^{x_1} F(y')dx$$

Show that the transversality is the same with orthogonality.
Here our

$$F(y') = \sqrt{1 + (y')^2}$$

Recall the transversality condition:

$$\Big((F - F_{y'}y') + F_{y'}Y_i'\Big)\Big|_1^2 = 0$$

So computing

$$F_{y'} = \frac{y'}{\sqrt{1 + (y')^2}}$$

Then:

$$\begin{aligned}
F - \frac{y'}{\sqrt{1 + (y')^2}}y' + \frac{y'}{\sqrt{1 + (y')^2}}Y_i' &= 0 \text{ for } i = 1, 2 \\
\frac{\sqrt{1 + (y')^2} - (y')^2 + y'Y_i'}{\sqrt{1 + (y')^2}} &= 0 \\
\frac{1 + y'Y_i'}{\sqrt{1 + (y')^2}} &= 0 \\
1 + y'Y_i' &= 0 \text{ for } i = 1, 2 \\
y' &= -\frac{1}{Y_i'} \text{ for } i = 1, 2
\end{aligned}$$

And because the Y_i' are the slopes of the boundary curve i, and the y' is the slope of the solution function, we see that the solution function is orthogonal to both the boundary curves.

Example 31. Show that transversality is the same as orthogonal if and only if $F(x, y, y')$ has the form $F = g(x, y)\sqrt{1 + (y')^2}$ with $g(x, y) \neq 0$ near the point of intersection.
See HW for proof.

1.4 Day 4

1.4.1 Variation problem with Integral Constraints - Isoperimetric Problem

Among all the closed curves of a given length l, find the curve enclosing the greatest area. Note that **isoperimeter** mean **with the same perimeter**.

Theorem 32. *Given the functional*

$$J[y] = \int_a^b F(x, y, y')dx$$

Let the admissible curves satisfy these conditions:

1. $y(a) = A$, $y(b) = B$
2. $k[y] = \int_a^b G(x, y, y')dx = l$ *where* $k[y]$ *is another functional.(this is the integral constraints).*

Let $J[y]$ *have an extremum for* $y = y(x)$ *subject to the conditions above. Then if* $y = y(x)$ *is* ***not*** *an extremal of* $k[y]$*, there exists a constant* λ *such that* $y = y(x)$ *is an extremal of the functional:*

$$\int_a^b F - \lambda G dx$$

i.e., $y = y(x)$ *will satisfy the (E-L) differential equation:*

$$F_y - \frac{d}{dx}F_{y'}\lambda(G_y - \frac{d}{dx}G_y) = 0$$

Remark 33. Recall the method of Lagrange multiplier: To find the max/min of $f(x)$ subject to the constraint $g(x) = 0$, we form the Lagrange function:

$$L(x, \lambda) = f(x) - \lambda g(x)$$

and then find the stationary points of L considered as a function of x and the Lagrange multiplier λ.

Proof. Let $\tilde{y} = y + \varepsilon_1\eta_1 + \varepsilon_2\eta_2$, where $\varepsilon_2\eta_2$ is the correction term, it makes sure that $\tilde{y}$ satisfying the isometric constraints, regardless of what $\varepsilon_1\eta_1$ is. Then:

$$\begin{aligned} J[\tilde{y}] &= \int_a^b F(x, y + \varepsilon_1\eta_1 + \varepsilon_2\eta_2, y' + \varepsilon_1\eta_1' + \varepsilon_2\eta_2')dx \\ &= \theta(\varepsilon_1, \varepsilon_2) \end{aligned}$$

And

$$\begin{aligned} K[\tilde{y}] &= \int_a^b G(x, y + \varepsilon_1\eta_1 + \varepsilon_2\eta_2, y' + \varepsilon_1\eta_1' + \varepsilon_2\eta_2')dx \\ &= \Phi(\varepsilon_1, \varepsilon_2) = l \end{aligned}$$

The original optimizing functionals under constraint is converted to a simpler problem: optimizing $\theta(\varepsilon_1, \varepsilon_2)$ under the constraint of $\Phi(\varepsilon_1, \varepsilon_2) = l$.

When $(\varepsilon_1, \varepsilon_2) = (0, 0)$ and $\tilde{y} = y$ gives the extreme value of $J[y]$. Using the method of Lagrange multiplier, we get:

$$\begin{aligned}\nabla\theta(\varepsilon_1, \varepsilon_2) = &= \lambda\nabla(\Phi(\varepsilon_1, \varepsilon_2) - l) \\ &= \lambda\nabla\Phi(\varepsilon_1, \varepsilon_2)\end{aligned}$$

Therefore:

$$\nabla\theta(\varepsilon_1, \varepsilon_2) - \lambda\nabla\Phi(\varepsilon_1, \varepsilon_2) = 0$$

Now the partial derivatives are:

$$\left.(\frac{\partial\theta}{\partial\varepsilon_1} - \lambda\frac{\partial}{\partial\varepsilon_1}\Phi)\right|_{\varepsilon=0} = 0$$

and

$$\left.(\frac{\partial\theta}{\partial\varepsilon_2} - \lambda\frac{\partial}{\partial\varepsilon_2}\Phi)\right|_{\varepsilon=0} = 0$$

Let's work on the first equation and take a look at the two terms:

1. the term:

$$\begin{aligned}\left.\frac{\partial\theta}{\partial\varepsilon_1}\right|_{\varepsilon=0} &= \left.\frac{\partial}{\partial\varepsilon_1}\int_a^b F(x, \tilde{y}, \tilde{y}')\right|_{\varepsilon=0} \\ &= \int_a^b \frac{\partial F}{\partial y}\eta_1 + \frac{\partial F}{\partial y'}\eta_1' dx \\ &= \int_a^b \left(\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'}\right)\eta_1 dx\end{aligned}$$

2. Similarly, the second term is now:

$$\left.\frac{\partial\Phi}{\partial\varepsilon_1}\right|_{\varepsilon=0} = \int_a^b \left(\frac{\partial G}{\partial y} - \frac{d}{dx}\frac{\partial G}{\partial y'}\right)\eta_1 dx$$

Therefore the first equation is now:

$$\int_a^b \left[(\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'}) - \lambda(\frac{\partial G}{\partial y} - \frac{d}{dx}\frac{\partial G}{\partial y'})\right]\eta_1 dx = 0$$

Since η_1 is arbitrary function, by the fundamental lemma of calculus of variation, we get the intergrand must be zero:

$$(F_y - \frac{\partial}{\partial x}F_{y'} - \lambda(G_y - \frac{\partial}{\partial x})G_{y'} = 0$$

And this completed the proof. □

Example 34 (Classical Dido's problem)**.** Among all curves of length l in the upper half plan passing through the points $(-a, 0)$ and $(a, 0)$, find the one which together with the interval $[-a, a]$ encloses the largest area.
We suspect that it should be a upper half circle. Let's prove it:
The arc length is the functional

$$J[y] = \int_{-a}^{a} y(x)dx$$

We wish to find $y(x)$ such that $J[y]$ takes the largest value subject to the conditions:

$$y(-a) = y(a) = 0, \text{ and } k[y] = \int_{-1}^{a} \sqrt{1 + (y')^2}dx = l$$

From the theorem we consider the functional:

$$J[y] - \lambda k[y] = \int_{-a}^{a} y - \lambda\sqrt{1 + (y')^2}dx$$

Let us denote $M = y - \lambda\sqrt{1 + (y')^2}$, then the corresponding E-L equation for M is:

$$\begin{aligned}
M_y - \frac{d}{dx}M_{y'} &= 0 \\
1 + \frac{d}{dx}(\lambda\frac{y'}{\sqrt{1 + (y')^2}}) &= 0 \\
x + \lambda\frac{y'}{\sqrt{1 + (y')^2}} &= C_1 \\
(C_1 - x)^2 &= \lambda^2\frac{(y')^2}{1 + (y')^2} \\
(y')^2(\lambda^2 - (x - C_1)^2) &= (C_1 - x)^2 \\
y' &= \pm\frac{C_1 - x}{\sqrt{\lambda^2 - (x - C_1)^2}}dx
\end{aligned}$$

Take the plus case. Let $x - C_1 = \lambda \sin\theta$, then $dx = \lambda \cos d\theta$. Also $C_1 - x = -\lambda \sin\theta$:

$$\begin{aligned}
y &= \int \frac{-\lambda\sin\theta}{\cos\theta}\lambda\cos\theta d\theta \\
&= -\lambda\int \sin\theta d\theta = -\lambda\cos\theta + C_2
\end{aligned}$$

Therefore

$$x - C_1 = \lambda\sin\theta$$

and

$$y - C_2 = \lambda\cos\theta$$

And so we get:

$$(x - C_1)^2 + (y - C_2)^2 = \lambda^2$$

Where the C_1, C_2, λ are determined by the constraints. (which you can figure out with the initial points $(-a, 0)$ and $(a, 0)$).

1.4.2 Holonomic Constraints - Pointwise constraints

Definition 35 (Holonomic Constraints)**.** A constraint on a dynamical system that can be integrated in such a way to eliminate one of the variables is called a holonomic constraint. A constraint that cannot be integrated is called a nonholonomic constraint.
The point is that a constraint is holonomic if it can be integrated (to give $g(q,t) = 0$) and then applied to reduce the degrees of freedom in a system.

Theorem 36. *Given the functional*

$$J[y,z] = \int_a^b F(x,y,z,y',z')dx$$

Let the admissible curves lie on the surface:
1. $g(x,y,z) = 0$
and satisfy the boundary conditions:

$$y(a) = A_1, y(b) = B_1$$

$$z(a) = A_2, a(b) = B_2$$

Suppose $J[y,z]$ *has an extremum for the curve:*
2. $y = y(x)$ *and* $z = z(x)$
Then if g_y *and* g_z *do not vanish simultaneously at any point of the surface 1, there exists a function* $\lambda(x)$ *such that 2 is an extremal of the functional:*

$$\int_a^b F + \lambda(x)g dx$$

i.e, it satisfies the differential equations:

$$F_y + \lambda g_y - \frac{d}{dx}F_{y'} = 0$$

$$F_z + \lambda g_z - \frac{d}{dx}F_{z'} = 0$$

Proof. Skipped. □

Example 37. Among all curves lying on the sphere $x^2 + y^2 + z^2 = a^2$ and passing through two given points (x_0, y_0, z_0) and (x_1, y_1, z_1), find the one which has the least length. (Finding the geodesics).
The length of the curve $x = x(t), y = y(t), z = z(t)$ between the two given points is:

$$\int_{t_0}^{t_1} \sqrt{x'^2 + y'^2 + z'^2}dt$$

And the constraint is $x^2 + y^2 + z^2 - a^2 = 0$.
The auxilary functional is then:

$$\int_{t_0}^{t_1} \sqrt{x'^2 + y'^2 + z'^2} - \lambda(t)(x^2 + y^2 + z^2 - a^2)dt = 0$$

And then we can have the corresponding E-L equations are.... In some cases, these are easy to solve. (In some cases, they are not).

1.5 Day 5:

This section follows Chapter 4 of Gelfand Fomin Calculus of Variation book.

1.5.1 Non-holonomic Constraints - constraints are given by DEs

Suppose we have a functional:

$$J[y] = \int_a^b F(x, y, y')dx$$

The corresponding E-L equation is $F_y - \frac{d}{dx}F_{y'} = 0$. This is a second order differential equation. We can follow some steps to have another approach to this:

1. Reduce the E-L equation: let $y_1 = y'$ so we have

$$\frac{dy}{dx} = y_1$$

and so

$$F_y - \frac{d}{dx}F_{y_1} = 0$$

2. Canonical variable: $p = F_{y'}$. Then we change from:

$$x, y, y' \to x, y, p$$
$$F(x, y, y') \to H(x, y, p)$$

where F and H are related by $H = -F + y'p$, and we regarded y' as a function of x, y, p.
We called p the momentum, H is the Hamilton corresponding to functional $J[y]$. Note that the canonical variables are x, y, p, H.

Question 38. How do we express the E-L equation in terms of the new canonical variables $x.y, p, H$?
With

$$H = -F + y'p$$

We calculate:

$$\begin{aligned} dH &= -dF + dy'p + dp.y', \text{ note } F = F(x, y, y') \\ &= -(\frac{\partial F}{\partial x}.dx + \frac{\partial F}{\partial y}.dy + \frac{\partial F}{\partial y'}dy') + dy'p + dp.y' \\ &= -\frac{\partial F}{\partial x}.dx - \frac{\partial F}{\partial y}.dy + y'dp, \text{ since } p = \frac{\partial F}{\partial y'}, \text{ the term } -\frac{\partial F}{\partial y'}dy' + dy'.p = 0 \end{aligned}$$

Now since $H = H(x, y, p)$, when taking derivatives we get:

$$dH = \frac{\partial H}{\partial x}dx + \frac{\partial H}{\partial y}dy + \frac{\partial H}{\partial p}dp$$

Then we compare the two equations and get:

$$-\frac{\partial F}{\partial x} = \frac{\partial H}{\partial x}, \ , \frac{\partial F}{\partial y} = -\frac{\partial H}{\partial y}, \ y' = \frac{\partial H}{\partial p}$$

So now we can write the E-L equation: $F_y - \frac{d}{dx}F_{y'} = 0$ in terms of H:

$$F_y = \frac{\partial F}{\partial y} = -\frac{\partial H}{\partial y}$$

and

$$F_{y'} = p$$

so the E-L equation is now:

$$-\frac{\partial H}{\partial y} - \frac{d}{dx}p = 0$$

which is equivalent to:

$$-\frac{\partial H}{\partial y} = \frac{dp}{dx}$$

Combining with the original set up of the canonical:

$$\frac{dy}{dx} = \frac{\partial H}{\partial p}$$

We have the following **Hamilton's system - Hamilton equations**:

$$\begin{cases} \frac{dy}{dx} = \frac{\partial H}{\partial p} \\ \frac{dp}{dx} = -\frac{\partial H}{\partial y} \end{cases} \tag{3}$$

Example 39. Consider $J[y] = \int_a^b \sqrt{1+(y')^2}dx$ the arc length functional. Then:

$$F = \sqrt{1+(y')^2}$$

And if we let

$$p = F_{y'} = \frac{y'}{\sqrt{1+(y')^2}}$$

We get:

$$(1+(y')^2)p^2 = (y')^2$$

And so:

$$y' = \frac{p}{\sqrt{1-p^2}}$$

Then we calculate the Hamiloton:

$$\begin{aligned} H(x,y,p) = -F + yp' &= -\sqrt{1+(y')^2} + \frac{p}{\sqrt{1-p^2}}.p \\ &= -\sqrt{1+\frac{p^2}{1-p^2}} + \frac{p^2}{\sqrt{1-p^2}} \\ &= -\frac{1}{\sqrt{1-p^2}} + \frac{p^2}{\sqrt{1-p^2}} \\ &= -\frac{1-p^2}{\sqrt{1-p^2}} \\ &= -\sqrt{1-p^2} \end{aligned}$$

The Hamilton system will be:

$$\begin{cases} \frac{dy}{dx} = \frac{\partial H}{\partial p} = \frac{p}{\sqrt{1-p^2}} \\ \frac{dp}{dx} = -\frac{\partial H}{\partial y} = 0 \end{cases}$$

Definition 40 (Variational Formulations)**.**

1. If $J = \int_{x_0}^{x_1} F dx$, then F is called the Lagrangian. Sometimes the Lagrangian is written as $\mathcal{L}$ instead of F.

2. As defined above, $H = H(x, y, p)$ is the Hamiltonian, we denote $\mathcal{L} = \mathcal{L}(t, q, \dot{q})$. Now sometimes we called **generalized coordinates** (t, q, p) in which $p = \dot{q}$.

3. The transformation $(q, \dot{q}) \iff (q, p)$ such that:

$$H = -\mathcal{L} + p.\dot{q}$$

is called the **Legendre transformation**. This transformation is **an involution**, i.e., a transformation which is its own inverse:

   (a) The transformation from $\mathcal{L}$ to H ($\mathcal{L} \to H$):

$$H = -\mathcal{L} + p\dot{q} = -\mathcal{L} + \frac{\partial \mathcal{L}}{\partial \dot{q}}.\dot{q}$$

   (b) The transformation from H to $\mathcal{L}$ ($H \to \mathcal{L}$):

$$\begin{aligned} -H + \frac{\partial H}{\partial p}.p &= -H + \dot{q}p \\ &= -(-\mathcal{L} + \frac{\partial \mathcal{L}}{\partial \dot{q}}.\dot{q}) + \dot{q}p \\ &= \mathcal{L} \text{ , since } p = \frac{\partial \mathcal{L}}{\partial \dot{q}} \end{aligned}$$

4. Now the functional J is:

$$J[p, q] = \int_a^b -H(t, q, p) + p\dot{q} dt$$

Where we also have the **Canonical Euler equations**:

$$\begin{cases} p : -\frac{\partial H}{\partial p} + \dot{q} = 0 \\ q : -\frac{\partial H}{\partial q} - \dot{p} = 0 \end{cases} \tag{4}$$

Definition 41 (Physical Interpretation of functional J)**.** Let $J[q] = \int_{t_0}^{t_1} \mathcal{L}(t, q, \dot{q}) dt$. Consider the motion of a single particle in $\mathbb{R}^3$. Let $\vec{q}(t) = (x(t), y(t), z(t))$ denote the position of the particle at time t. Suppose it has mass m and let $\dot{} = \frac{d}{dt}$ (dot over a quantity denote the derivative in time). Then the kinetic energy of the particle is:

$$T = \frac{1}{2} m(\dot{x}^2(t) + \dot{y}^2(t) + \dot{z}^2(t))$$

Assume there is a function $V = V(t, \vec{q})$, and the force $\vec{f} = (f_1, f_2, f_3)$ acting on the particle has the components $f_1 = -\frac{\partial V}{\partial x}$, $f_2 = -\frac{\partial V}{\partial y}$, $f_3 = -\frac{\partial V}{\partial z}$. Also, this V is called the potential function.
Denote $\mathcal{L} = T - V$, and this $\mathcal{L}$ is called the Lagrangian of this system.

Theorem 42 (Hamilton's Principle). *The motion of a system of particles $\vec{q}$ from a given initial configuration $\vec{q}(t_0)$ to a given final configuration $\vec{q}(t_1)$ in the time interval $[t_0, t_1]$ is such that the functional:*

$$J[\vec{q}] = \int_{t_0}^{t_1} \mathcal{L}(t, \vec{q}, \dot{\vec{q}})dt$$

is ***stationary.***

Remark 43. In the above, let's write:

$$\mathcal{L} = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(x, y, z)$$

For $u = x, y, z$, the E-L equation is:

$$\frac{\partial \mathcal{L}}{\partial u} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{u}} = 0$$
$$-\frac{\partial V}{\partial u} - m\ddot{u} = 0$$

Therefore we get:

$$m\ddot{q} = -\nabla V = f \text{ (force)}$$

where $f = m\vec{a}$: Newton's second law in three dimension.

Example 44 (Simple Pendulum). Consider a simple Pendulum of mass m and length l in the plane. Let $(x(t), y(t))$ denote the position of the mass at time t. As $x^2(t)+y^2(t) = l^2$, we only need one position variable, so we use polar coordinates instead and the position of the mass is characterized by the angle $\Phi(t)$ between the vertical and the string where the mass is attached.(See figure (7)) Then find the equation of the motion of the mass. We have the following:

1. The kinetic energy is $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}ml^2\dot{\Phi}^2(t)$
2. Potential energy: $V = mgh = mgl(1 - \cos\Phi(t))$

Then the Lagrangian is:

$$\mathcal{L}(\Phi, \dot{\Phi}) = \frac{1}{2}ml^2\dot{\Phi}^2 - mgl(1 - \cos\Phi)$$

The Hamilton principle implies that the motion from a given angle $\Phi(t_0)$ to a fixed angle $\Phi(t_1)$ is such that the functional:

$$J[\Phi] = \int_{t_0}^{t_1} \left(\frac{1}{2}ml^2\dot{\Phi}^2 - mgl(1 - \cos\Phi)\right)dt \textbf{ is stationary}$$

The E-L equation is (letting $q = \Phi$):

$$\frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} = -mgl\sin q - \frac{d}{dt}ml^2\dot{q}^2$$
$$= -mgl\sin q - ml^2\ddot{q} = 0$$

Therefore

$$\ddot{q} + \frac{g}{l}\sin q = 0 \text{ (equation of motion } F = ma)$$

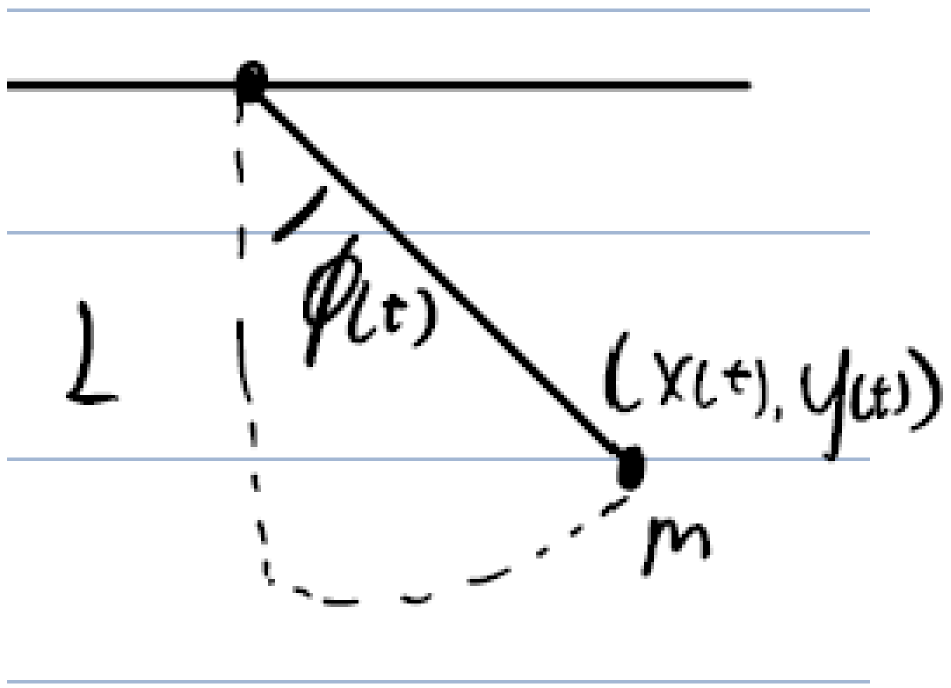

Figure 7: Simple pendulum

The general Hamilton theory (when transforming from $\mathcal{L}$ to H) is:

$$\begin{aligned} H &= -\mathcal{L} + q.\dot{q} \\ &= -T + V + m\dot{q}.\dot{q} \text{ since the momentum } p = \frac{\partial \mathcal{L}}{\partial \dot{q}} = m\dot{q} \\ &= -T + V + 2T = T + V \text{ (total energy of the system)} \end{aligned}$$

In the case of Simple Pendulum, we have the momentum:

$$p = \frac{\partial L}{\partial \dot{q}} = ml^2\dot{q}$$

Therefore:

$$\dot{q} = \frac{p}{ml^2}$$

Plugging it back we get:

$$\begin{aligned} H(t, q, p) &= -\mathcal{L} + p\dot{q} \\ &= -\frac{1}{2}ml^2(\frac{p}{ml^2})^2 + mgl(1 - \cos q) + p.\frac{p}{ml^2} \\ &= \frac{1}{2}\frac{p^2}{ml^2} + mgl(1 - \cos q) = T + V \end{aligned}$$

The Hamilton equations are:

$$\begin{cases} \dot{q} = \frac{\partial H}{\partial p} = \frac{p}{ml^2} \\ \dot{p} = -\frac{\partial H}{\partial q} = -mgl\sin q \end{cases}$$

Therefore, we get the E-L equation:

$$\ddot{q} = \frac{\dot{p}}{ml^2} = \frac{-mgl\sin q}{ml^2} = -\frac{g}{l}\sin q$$

The $H(t, q, p)$ is the total energy, when is the total energy conserved?

Theorem 45. *If the Hamiltonian function does not depend on t explicitly, then it is constant on any solution $(q(t), p(t))$ to the Hamilton's equations, i.e., $H(q, p) = C$ constant.*

Proof. Just need to show $\frac{dH}{dt} = 0$. In fact, we have:

$$\begin{aligned}\frac{d}{dt}H(q(t), p(t)) &= \frac{\partial H}{\partial q}\frac{dq}{dt} + \frac{\partial H}{\partial p}\frac{dp}{dt} \\ &= \frac{\partial H}{\partial q}\frac{\partial H}{\partial p} + \frac{\partial H}{\partial q}(-\frac{\partial H}{\partial q}) \text{ use Canonical Euler equation 4 } \dot{q} = \frac{\partial H}{\partial p}, \dot{p} = -\frac{\partial H}{\partial q} \\ &= 0\end{aligned}$$

Therefore $H(q, p)$ is a constant on (q, p). □

Definition 46 (First Integral)**.** The first integral of a system of differential equations is a function which has a constant value along each integral curve of the system.

Remark 47. Therefore, by previous theorem, if $\mathcal{L}$ does not depend on t explicitly, the function $H(q, p)$ is a first integral of the E-L equation.

1.6 Day 6:

1.6.1 Hamilton-Jacobi equation

Previously, we were given $J[y] = \int_{x_0}^{x_1} F(x, y, y')dx$, we find the extremal y. Now if we known y is the extremal, we evaluate the integral $\int_{x_0}^{x_1} F(x, y, y')dx = S$. We have this definition:

Definition 48 (action function)**.** The action function S is defined as the integral:

$$S = \int_{x_0}^{x_1} F(\xi, y, y')d\xi$$

evaluated along the extremal joining the points $A(x_0, y_0)$ and $B(x, y)$.
Note that $S = S(x, y)$ is a single-valued function of the coordinates of the point B.(see figure (8) for pictorial reference $B_1, B_2, \ldots$)

Remark 49. With the definition of action function, we consider the 2 ways to evaluate its derivatives:

1. The first way:

$$\frac{dS}{dx} = \frac{\partial S}{\partial y}y' + \frac{\partial S}{\partial x} \tag{5}$$

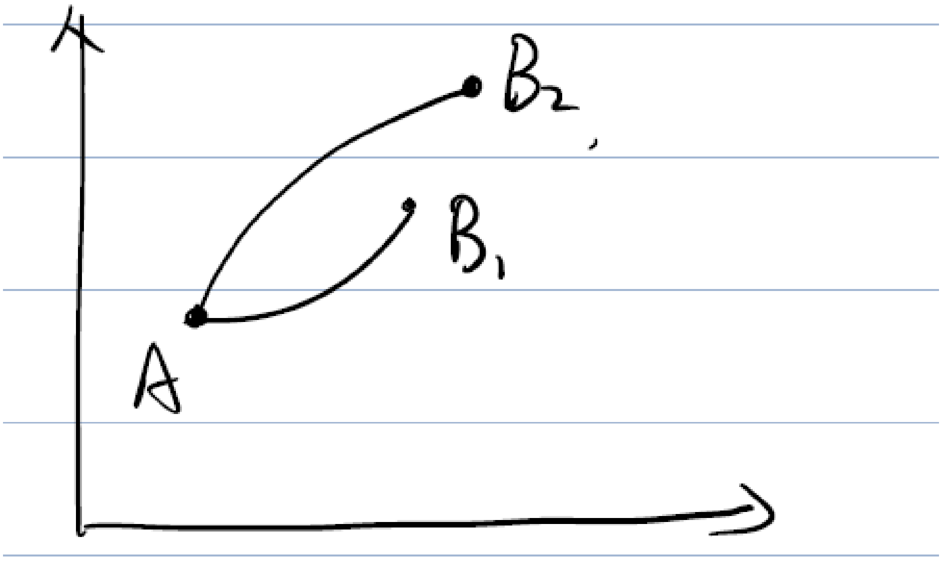

Figure 8: Hamilton Jacobian picture

2. the second way: Recall that
$$H = -F + py'$$
Therefore:
$$F = -H + py'$$
So
$$S = \int_{x_0}^{x_1} F(\xi, y, y') = \int_{x+0}^{x_1} -H + py' d\xi$$
Therefore:
$$-\frac{dS}{dx} = -H + py' \tag{6}$$

If we compare (5) and (6) we get:
$$\begin{cases} p = \frac{\partial S}{\partial y} \\ -H = \frac{\partial S}{\partial x} \end{cases} \tag{7}$$
In particular, we get:
$$\frac{\partial S}{\partial x} + H(x, y, p) = 0$$
And we call this **Hamilton-Jacobi equation**:
$$\frac{\partial S}{\partial x} + H(x, y, \frac{\partial S}{\partial y}) = 0, \text{ where } S = \int_{C_0} pdy - Hdx \text{ for } C_0 \text{ an extremal} \tag{8}$$

Example 50. For a functional $J[y] = \int_a^b (y')^2 dx$, we want to find the H-J equation:
$F = (y')^2$ and $H = -F + py'$.
Now:
$$p = \frac{\partial F}{\partial y'} = 2y'$$

and

$$y' = \frac{1}{2}p$$

Then

$$H = -(\frac{1}{2}p)^2 + p.\frac{1}{2}p = \frac{1}{4}p^2$$

So the Hamilton-Jacobi equation is:

$$\frac{\partial S}{\partial x} + \frac{1}{4}(\frac{\partial S}{\partial y})^2 = 0$$

Example 51. Consider the light propagation in some medium. The action function is the total travel time:

$$S = \int_A^B n(x,y)\sqrt{1+(y')^2}dx$$

Denote

$$\mathcal{L} = n(x,y)\sqrt{1+(y')^2}$$

Then:

$$p = \frac{\partial \mathcal{L}}{\partial y'} = n(x,y)\frac{y'}{\sqrt{1+(y')^2}}$$

Now we can solve for y' from this equation and get:

$$y' = \frac{p}{\sqrt{n^2-p^2}}$$

Then we calculate H:

$$\begin{aligned} H &= -\mathcal{L} + py' = -n(x,y)\sqrt{1+(y')^2} + p\frac{p}{\sqrt{n^2-p^2}} \\ &= -\sqrt{n^2-p^2} \end{aligned}$$

And so the H-J equation is:

$$\begin{gathered} \frac{\partial S}{\partial x} + H(x,y,\frac{\partial S}{\partial y}) = 0 \\ \frac{\partial S}{\partial x} + (-\sqrt{n^2 - \frac{\partial S}{\partial y}^2}) = 0 \\ (\frac{\partial S}{\partial x})^2 + (\frac{\partial S}{\partial y})^2 = n^2 : \text{Eikonal equation} \\ (\nabla S)^2 = n^2 \end{gathered}$$

Question 52. We have some follow-up questions:

1. How to solve the H-J equation for S?

2. How is the solution to H-J related to the extremals?

We have some answers to the above questions, for some special cases:

Remark 53. The special cases are when H does not depends on x or when H does not depend on y:

1. If $\frac{\partial H}{\partial x} = 0$, that is, $H = H(y, p)$. Then $H =$ constant on extremal C_0. So suppose $H = \alpha$. Then:
$$\frac{\partial S}{\partial x} = -\alpha$$
So
$$S(x, y) = S_0(y) - \alpha x$$
2. If $\frac{\partial H}{\partial y} = 0$, i.e, $H = H(x, p)$. What is the expression of S? Since
$$\frac{dp}{dx} = -\frac{\partial H}{\partial y} = 0$$
So $p = C$ constant on C_0. Then:
$$\begin{aligned} S &= \int_{C_0} pdy - Hdx \\ &= \int_{C_0} Cdy - H(x, C)dx \\ &= Cy - \int_{C_0} H(x, C)dx \end{aligned}$$

1.6.2 n-dimensional Hamilton-Jacobi equation

Consider the general n-dimensional case of the H-J equation:
$$\frac{\partial S}{\partial x} + H(x, y_1, \ldots, y_n, \frac{\partial S}{\partial y_1}, \ldots, \frac{\partial S}{\partial y_n}) = 0 \tag{9}$$

Theorem 54. *Let*
$$S = S(x, y_1, \ldots, y_n, \alpha_1, \ldots, \alpha_n)$$
be a solution (to the H-J equation), depending on $m(\leq n)$ parameters $\alpha_1, \ldots, \alpha_m$ of the H-J equation (9). Then each derivative:
$$\frac{\partial S}{\partial \alpha_i}, \ \textit{for } i = 1, 2, \ldots, m$$
is a first integral of the system of Canonical Euler equations:
$$\frac{dy_i}{dx} = \frac{\partial H}{\partial p_i}, \ \frac{dp_i}{dx} = -\frac{\partial H}{\partial y_i}$$
i.e., $\frac{\partial S}{\partial \alpha_i} =$ const, along each extremal $i = 1, 2, \ldots, m$.

Proof. Only need to show that $\frac{\partial}{dx}(\frac{\partial S}{\partial \alpha_i}) = 0$ when y_i and p_i satisfy the Hamilton's equation. We get:
$$\frac{\partial}{dx}(\frac{\partial S}{\partial \alpha_i}) = \frac{\partial^2 S}{\partial x \partial \alpha_i} + \sum_{k=1}^{n} \frac{\partial^2 S}{\partial y_k \partial \alpha_i} \frac{dy_k}{dx}$$
From the H-J equation:
$$\frac{\partial S}{\partial x} = -H(x, y_1, \ldots, y_n, \frac{\partial S}{\partial y_1}, \ldots, \frac{\partial S}{\partial y_n})$$

So taking partial derivative of this again wrt the α_i give:

$$\frac{\partial^2 S}{\partial x \partial \alpha_i} = -\sum_{k=1}^{n} \frac{\partial H}{\partial p_k} \frac{\partial^2 S}{\partial y_k \partial \alpha_i}$$

Therefore combining we get:

$$\begin{aligned} \frac{\partial}{dx}\left(\frac{\partial S}{\partial \alpha_i}\right) &= \sum_{k=1}^{n} \left[\frac{\partial^2 S}{\partial y_k \partial \alpha_i} \left(\frac{dy_k}{dx} - \frac{\partial H}{\partial p_k} \right) \right] \\ &= 0 \end{aligned}$$

Since along each extremal, we get $\frac{dy_k}{dx} - \frac{\partial H}{\partial p_k} = 0$. Therefore:

$$\frac{\partial S}{\partial \alpha_i} = \beta_i = \text{constant}$$

along each extremal. □

Theorem 55. *Let $S = S(x, y_1, \ldots, y_n, \alpha_1, \ldots, \alpha_n)$ be a general solution of the H-J equation (9) depending on n parameters $\alpha_1, \ldots, \alpha_n$. Let the determinant of the $n \times n$ matrix:*

$$\left\| \frac{\partial^2 S}{\partial \alpha_i \partial y_k} \right\| \neq 0$$

Let $\beta_1, \ldots, \beta_n$ be n arbitrary constants. Then the functions

$$y_i = y_i(x, \alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n)$$

defined by the relations:

$$\frac{\partial}{\partial \alpha_i} S(x, y_1, \ldots, y_n, \alpha_1, \ldots, \alpha_n) = \beta_i, \ \textit{for } i = 1, 2, \ldots, n$$

together with the functions:

$$p_i = \frac{\partial}{\partial y_i} S(x, y_1, \ldots, y_n, \alpha_1, \ldots, \alpha_n)$$

with y_i given above, will constitute a general solution of the Hamilton system:

$$\frac{dy_i}{dx} = \frac{dH}{dp_i}, \ \frac{dp_i}{dy_i} = -\frac{\partial H}{\partial y_i} \ \ i = 1, 2, \ldots, n$$

Proof. Recall that:

$$\frac{\partial}{dx}\left(\frac{\partial S}{\partial \alpha_i}\right) = \frac{\partial^2 S}{\partial x \partial \alpha_i} + \sum_{k=1}^{n} \frac{\partial^2 S}{\partial y_k \partial \alpha_i} \frac{dy_k}{dx}$$

As in the previous theorem, we also have:

$$\frac{\partial^2 S}{\partial x \partial \alpha_i} = -\sum_{k=1}^{n} \frac{\partial H}{\partial p_k} \frac{\partial^2 S}{\partial y_k \partial \alpha_i} \text{ from the H-J equation}$$

Therefore:
$$\frac{\partial}{dx}(\frac{\partial S}{\partial \alpha_i}) = \sum_{k=1}^{n}\left[\frac{\partial^2 S}{\partial y_k \partial \alpha_i}\left(\frac{dy_k}{dx} - \frac{\partial H}{\partial p_k}\right)\right]$$
But since
$$\frac{\partial}{\partial \alpha_i}S(x, y_1, \ldots, y_n, \alpha_1, \ldots, \alpha_n) = \beta_i, \text{ for } i = 1, 2, \ldots, n$$
So
$$\frac{\partial}{dx}(\frac{\partial S}{\partial \alpha_i}) = 0 \text{ for } i = 1, 2, \ldots, n$$
Thus we must have:
$$\frac{dy_i}{dx} = \frac{dH}{dp_i} \text{ for } i = 1, 2, \ldots, n$$
Now for the other, first note that:
$$\begin{aligned}\frac{dp_i}{dx} = \frac{d}{dx}(\frac{\partial S}{\partial y_i}) &= \frac{\partial^2 S}{\partial x \partial y_i} + \sum_{k=1}^{n}\frac{\partial^2 S}{\partial y_k \partial y_i}\frac{dy_k}{dx} \\ &= \frac{\partial^2 S}{\partial x \partial y_i} + \sum_{k=1}^{n}\frac{\partial^2 S}{\partial y_k \partial y_i}\frac{dH}{dp_k} \text{ , use } \frac{dy_k}{dx} = \frac{dH}{dp_k}\end{aligned}$$

In short:
$$\frac{dp_i}{dx} = \frac{\partial^2 S}{\partial x \partial y_i} + \sum_{k=1}^{n}\frac{\partial^2 S}{\partial y_k \partial y_i}\frac{dH}{dp_k} \tag{10}$$
On the other hand, partial derivative of H-J equation $\frac{\partial S}{\partial x} = -H(x, y, \frac{\partial S}{\partial y})$ wrt y_i gives:
$$\frac{\partial^2 S}{\partial x \partial y_i} = -\frac{\partial H}{\partial y_i} - \sum_{k=1}^{n}\frac{dH}{dp_k}\frac{\partial^2 S}{\partial y_k \partial y_i}$$
Or equivalently:
$$-\frac{\partial H}{\partial y_i} = \frac{\partial^2 S}{\partial x \partial y_i} + \sum_{k=1}^{n}\frac{dH}{dp_k}\frac{\partial^2 S}{\partial y_k \partial y_i} \tag{11}$$
Comparing the equation (10) and (11) give:
$$\frac{dp_i}{dy_i} = -\frac{\partial H}{\partial y_i}$$
And we have shown both parts. □

1.6.3 A summary of different formulations of variational problem (1 unknown function)

There are 3 ways:

1. E-L equations: which is a second order DEs, given by:
$$\frac{dF}{dy} - \frac{d}{dx}\frac{dF}{dy'} = 0$$

2. The Hamilton's equations: first order DEs:

$$\begin{cases} \frac{dy}{dx} = \frac{\partial H}{\partial p} \\ \frac{dp}{dx} = -\frac{\partial H}{\partial y} \end{cases}$$

3. The Hamilton-Jacobi equation, which is first order non-linear PDE:

$$\frac{\partial S}{\partial x} = -H(x, y, \frac{\partial S}{\partial y})$$

Example 56. Recall an earlier example, $J[y] = \int_a^b (y')^2 dx$.

1. The H-J equation: In this example we have $F = (y')^2$ and $H = \frac{1}{4}p^2$. Also recall that this example has the H-J equation:

$$\frac{\partial S}{\partial x} + \frac{1}{4}(\frac{\partial S}{\partial y})^2 = 0$$

Let's solve for y in this H-J equation. Recall earlier a result that: if $\frac{\partial H}{\partial x} = 0$, then $S(x, y) = S_0(y) - \alpha x$. Applying the H-J equation to this we get:

$$\begin{aligned} -\alpha + \frac{1}{4}(S_0'(y))^2 &= 0 \\ (S_0'(y))^2 &= 4\alpha \\ S_0'(y) &= \pm 2\sqrt{\alpha} \\ S_0(y) &= \pm 2\sqrt{\alpha}y \text{ , use separable technique} \\ S(x, y) &= \pm 2\sqrt{\alpha}y - \alpha x \end{aligned}$$

Take the plus case:

$$S(x, y) = 2\sqrt{\alpha}y - \alpha x$$

Then since

$$\frac{\partial S}{\partial y} = p \Rightarrow p = 2\sqrt{\alpha}$$

And:

$$\frac{\partial S}{\partial \alpha} = \beta \Rightarrow 2\frac{1}{2}\frac{1}{\sqrt{\alpha}}y - x = \beta \Rightarrow y = \sqrt{\alpha}x + \sqrt{\alpha}\beta$$

2. E-L equation:

$$F_y - \frac{d}{dx}F_{y'} = 0$$

This gives

$$y'' = 0 \Rightarrow y = mx + b$$

3. Hamilton's equation:

$$H = -F + py' = \frac{1}{4}p^2$$

Then:

$$\frac{dy}{dx} = \frac{\partial H}{\partial p} = \frac{1}{2}p$$

This gives

$$y = \frac{1}{2}px + d$$

In addition, the second part of Hamilton equation:

$$\frac{dp}{dx} = -\frac{\partial H}{\partial y} = 0$$

This gives $p = c$, a constant. Hence our solution is

$$y = \frac{1}{2}cx + d$$

2 Discrete dynamical systems

2.1 Day 7: Dynamical system

2.1.1 Introduction: Dynamical system definitions

We begin with some elementary property of homeomorphism function.

Definition 57 (homeomorphism)**.** Let I and J be intervals and $f : I \to J$. The function $f(x)$ is a homeomorphism if $f(x)$ is one-to-one, onto, and continuous, and $f^{-1}(x)$ is also continuous.

Definition 58 (C^r-diffeomorphism)**.** Let $f : I \to J$. The function $f(x)$ is a C^r-diffeomorphism if $f(x)$ is a C^r-homeomorphism such that $f^{-1}(x)$ is also C^r.

Example 59.

1. $\tan x : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$. This function is homeomorphism and also C^∞-diffeomorphism.
2. $f(x) = x^3 : \mathbb{R} \to \mathbb{R}$ is homeomorphism. It is **not** a diffeomorphism as $f^{-1}(x) = x^{1/3}$. And so $(f^{-1}(x))' = \frac{1}{3}x^{-\frac{2}{3}}$. Then $(f^{-1})'(0)$ does not exists.

Remark 60. If $f(x)$ is a homeomorphism from $\mathbb{R} \to \mathbb{R}$, then f is strictly monotonic. The proof is HW.

Definition 61 (Dynamic)**. Dynamic** means evolves with time.

Definition 62 (Discrete Dynamical system)**. Discrete dynamical system** occurs when time is considered to be discrete rather than continuous. A 1-dimension discrete time system has the form:

$$x_{n+1} = F(x_n) \text{ for } n = 0, 1, 2, \dots \text{ with } x_0 \text{ given.}$$

Example 63. We will discuss these models in detail later:

1. Logistic (growth) model
2. Newton's method: $x_{n+1} = x_n - \frac{F(x_n)}{F'(x_n)}$

Example 64 (Logistic Model)**.** Let p be the population size. The natural growth is that:

$$\frac{dp}{dt} = kp$$

where the k depends on p as well, and can be written as $k(p) = a - bp$ for $a, b > 0$. Then:

$$\frac{dp}{dt} = k(p)p = (a - bp)p = ap - bp^2$$

As an exercise, we can solve this system analytically:

$$\begin{cases} \frac{dp}{dt} = k(p)p = (a - bp)p = ap - bp^2 \\ p(0) = p_0 \end{cases}$$

Example 65 (Logistic growth - Euler's method)**.** We consider a set of discrete time $\{t_0, t_1, \ldots, t_n, \ldots\}$ where $t_n = t_0 + nh$ (or $t_n = t_{n-1} + h$). We also set $p_0 = p(t_0)$. We then can try to approximate $p(t)$ at time t_i using Euler's method (writing $p_n = p(t_n)$):

$$\frac{dp(t_n)}{dt} = \frac{p_{n+1} - p_n}{h(t_{n+1} - t_n)} = p_n(a - bp_n)$$

Multiply it all out we get:

$$p_{n+1} = p_n + (ap_n - bp_n^2)h$$

If we define $F(p) = p + (ap - bp^2)h$, then we have a discrete system $p_{n+1} = F(p_n)$. This is a good time to first see the idea of **canonical form of logistic growth**:

$$p_{n+1} = p_n(1 + ah) - p_n^2 bh$$

Letting $r = 1 + ah$ and $s = bh$, we have:

$$p_{n+1} = rp_n - sp_n^2$$

And if we let $x_n = \frac{s}{r} p_n$ we get the **canonical form**:

$$x_{n+1} = rx_n(1 - x_n)$$

2.1.2 Fixed points and orbits

Definition 66. We have these important related definitions in studying a discrete dynamical system:

1. Let $x_{n+1} = F(x_n)$ be a discrete dynamical system.
2. The $\{x_0, x_1, \ldots\}$ is called the **trajectory of the system**.
3. The function F that gives such system are called **mapping** or **map**.
4. The **forward orbit of** x is another name for the trajectory, denoted:
$$O_F^+(x) = \{x, F(x), F^2(x), \ldots\}$$
5. If F is a homeomorphism, then the **backward orbit of** x is
$$O_F^-(x) = \{x, F^{-1}(x), F^{-2}(x), \ldots\}$$

6. the **orbit of** x is $O(x) := O^+ \cup O^-$
7. x is a **fixed point** of F if $F(x) = x$.
8. A point x is **periodic** if there exists a n such that $F^n(x) = x$. The **prime period** of x is defined to be the smallest such n.
9. As a notation, we denote: $\text{Per}_n(F) = \{F^n(x) = x\}$ and $\text{Fix}(F) = \text{Per}_1(F)$.

Example 67. For $F(x) = 1 - x^2$. What is Fix(F)?

$$\begin{aligned} x &= 1 - x^2 \\ x^2 + x - 1 &= 0 \\ x &= -\frac{1}{2} \pm \frac{\sqrt{5}}{2} \end{aligned}$$

Therefore

$$\text{Fix}(F) = \{-\frac{1}{2} + \frac{\sqrt{5}}{2}, -\frac{1}{2} - \frac{\sqrt{5}}{2}\}$$

What are the periodic points of period 2?

$$\begin{aligned} x &= F(F(x)) = 1 - (F(x))^2 \\ &= 1 - (1 - (1 - x^2))^2 \\ &= 2x^2 - x^4 \end{aligned}$$

Therefore we solve:

$$\begin{aligned} x &= 2x^2 - x^4 \\ x(x^3 - 2x + 1) &= 0 \\ x(x-1)(x^2 + x - 1) &= 0 \\ x &= 0, 1, -\frac{1}{2} \pm \frac{\sqrt{5}}{2} \end{aligned}$$

Check $x = 0, F(0) = 1$ and $x = 1, F(1) = 0$. We also observe that: if $x^* = -\frac{1}{2} \pm \frac{\sqrt{5}}{2}$ satisfies $x^* = F(x^*)$, then $F(x^*) = F(F(x^*)) = F^2(x^*)$. In order words, if $x^* = F(x^*)$, then $x^* = F^2(x^*)$. (A fixed point will also be a periodic n-point).

Question 68. Will we always be able to find fixed point?

Proposition 69. Let $I = [a, b]$ be an interval and $f : I \to J$ be continuous. Then f has at least one fixed point in I.

Proof. Let $g(x) = f(x) - x$. Then $g(x)$ is also continuous on I. Suppose $f(a) > a$ and $f(b) < b$ (otherwise, one of a or b is fixed and we are done). Then $g(a) = f(a) - a > 0$ and $g(b) = f(b) - b < 0$. From IVT, there exists a c between a and b such that $g(c) = 0$. Then at c, $f(c) = c$ so f has at least one fixed point in I. $\square$

Definition 70 (eventually periodic)**.** A point x is **eventually periodic of period** n if x is not periodic but there exists $m > 0$ such that $F^{n+i}(x) = F^i(x)$ for all $i \geq m$. That is, $F^i(x)$ is periodic for $i \geq m$.

Example 71. The function $f(x) = x^2$ has fixed points $x = 0, 1$ satisfying $x^2 = x$. In addition, when $x = -1$, $f(-1) = 1$ so -1 is eventually fixed.

Example 72. Let S^1 denote the unit circle on the plane. We denote a point by its angle θ measured in radians. Let $f(\theta) = 2\theta$. Then we see that $f(\theta + 2\pi) = f(\theta)$ on the circle so the map is well-defined. Note also that $f^n(\theta) = 2^n\theta$. We see that: θ is periodic of period n iff $2^n\theta = \theta + 2k\pi$ for some integer k,i.e., iff $\theta = \frac{2k\pi}{2^n-1}$ where $0 \leq k \leq 2^n$ is an integer. In conclusion: $f(0) = 0$ is fixed. If $\theta = \frac{2k\pi}{2^n}$, then $f^n(\theta) = 2k\pi$ so that θ is eventually fixed.

Remark 73. Eventually periodic points cannot occur if the map is a homeomorphism.

2.2 Day 8: fixed point analysis

2.2.1 Asymptotic definition

Definition 74 (forward asymptotic)**.** A point x is **forward asymptotic** to p if

$$|f^i(x) - f^i(p)| \to 0 \text{ as } i \to \infty$$

Definition 75 (backward asymptotic)**.** A point x is **backward asymptotic** to p if

$$|f^{-i}(x) - f^{-i}(p)| \to 0 \text{ as } i \to \infty$$

Definition 76 (forward asymptotic and backward asymptotic of periodic point)**.** Let p be a periodic of period n.
A point x is forward asymptotic to p if $\lim_{i\to\infty} F^{in}(x) = p$.
The **stable set** of p is:

$$W^s(p) := \{q : q \text{ is forward asymptotic to } p\}$$

If F is invertible, then x is **backward asymptotic of** p if $\lim_{i\to\infty} F^{-in}(x) = p$.
The **unstable set of** p is:

$$W^u(p) := \{q : q \text{ is backward asymptotic to } p\}$$

Example 77. Let $f(x) = x^3$. then $f^{-1}(x) = x^{\frac{1}{3}}$. The fixed points of f are $x^3 = x$ so $x = 0, \pm 1$. We get:

$$W^s(0) = (-1, 1)$$

since for all $y \in (-1, 1)$: the limit $\lim_{i\to\infty} f(y) = \lim_{i\to\infty} y^{3i} = 0$.
And:

$$W^u(1) = \text{ positive real axis}$$

since for all y in positive real axis, the limit: $\lim_{i\to\infty} f^{-i}(y) = \lim_{i\to\infty} y^{\frac{1}{3i}} = 1$, and:

$$W^u(-1) = \text{ negative real axis}$$

since for all y in negative real axis, the limit: $\lim_{i\to\infty} f^{-i}(y) = \lim_{i\to\infty} y^{\frac{1}{3i}} = -1$.

Definition 78 (critical point)**.** A point x is **a critical point** of F if $F'(x) = 0$. The critical point is **non-degenerate** if $F''(x) \neq 0$, and is **degenerate** otherwise.

Example 79. We see some examples:

1. For $F(x) = x^2$, then $F'(x) = 2x$, so $F'(x) = 0$ when $x = 0$. So $x = 0$ is a critical point. In addition, $F''(x) = 2 \neq 0$, so 0 is a non-degenerate critical point.

2. $F(x) = x^n$. If $n > 2$, then $F(x)$ has a degenerate critical point at $x = 0$.

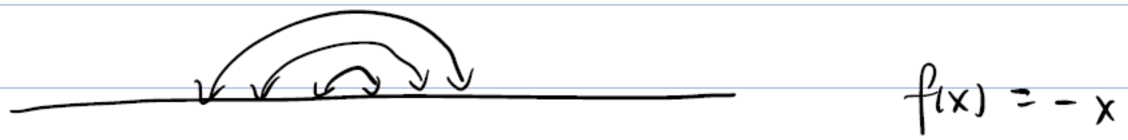

Figure 9: Phase Portrait $f(x) = -x$

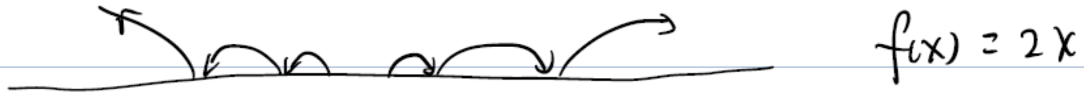

Figure 10: Phase Portrait $f(x) = 2x$

2.2.2 Phase portrait and Cobweb diagram

Definition 80 (Phase Portrait)**.** The **Phase Portrait** is a picture on the real line of all the orbits (recall orbit is the union of forward orbit $\{x, F(x), F^2(x), \ldots\}$ and backward orbit $\{x, F^{-1}(x), \ldots\}$).

Example 81. See figure (9) for $f(x) = -x$ and figure (10) for $f(x) = 2x$.

Definition 82 (Cobweb diagram)**.** Starting from an initial point x_0, draw a vertical line up to $f(x_0)$. From this point, draw a horizontal line either left or right to join the diagonal $y = x$. The x-coordinate is $x_1 = f(x_0)$. From the point $(x_1, f(x_0))$ draw a vertical line up or down to join $F(x)$, then draw a horizontal line to the diagonal, etc... See figure (11) for pictorial.

Example 83. Consider $x_{n+1} = \sqrt{x_n}$. The fixed points are $\sqrt{x^*} = x^*$ gives $x^* = 0, 1$. See (12) for Cobweb diagram. Observe that $x_n \to 1$ as $n \to \infty$, when $x_0 \in (0, 1)$. And similarly $x_n \to 1$ as $n \to \infty$, when $x_0 \in (1, \infty)$. So the convergence is monotonic.

Example 84. Consider $f(x) = \cos x$ so $x_{n+1} = \cos x_n$. See (13) for Cobweb diagram. In this case we see that x_0 converges to x^* through damped oscillations.

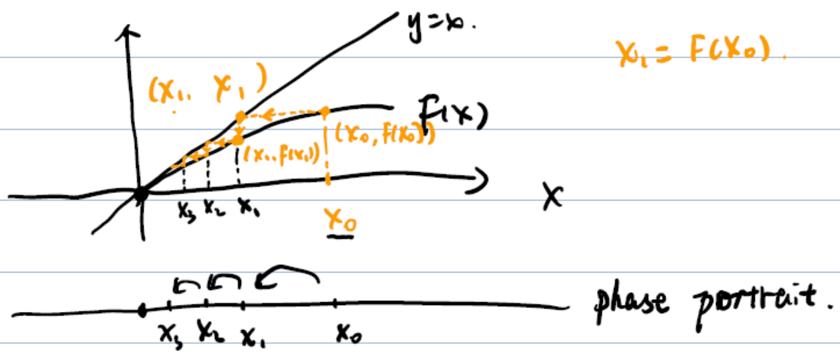

Figure 11: Cobweb and phase portrait

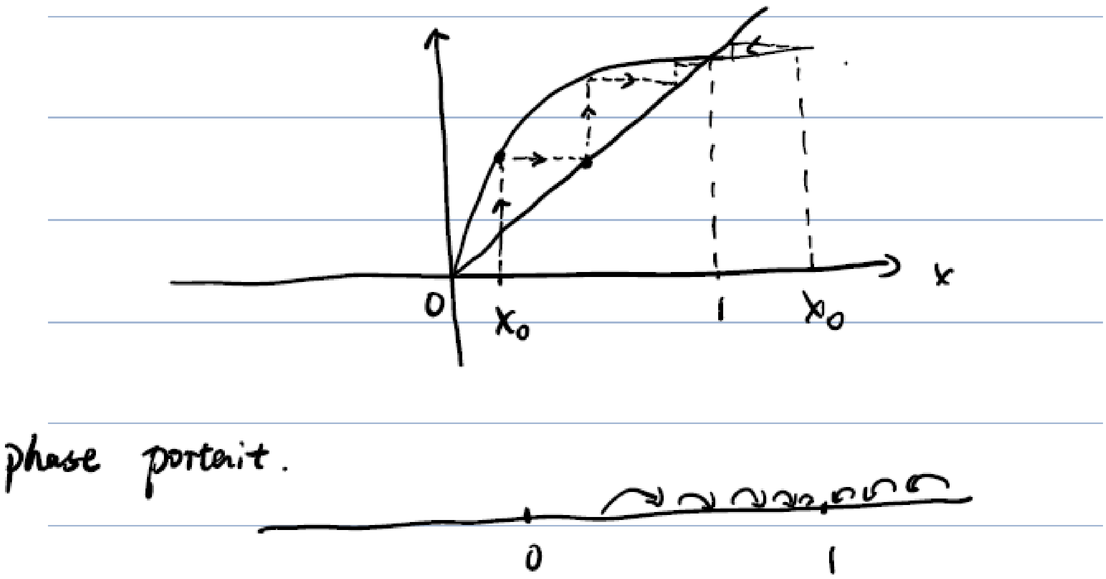

Figure 12: Cobweb and phase portrait for $x_{n+1} = \sqrt{x_n}$

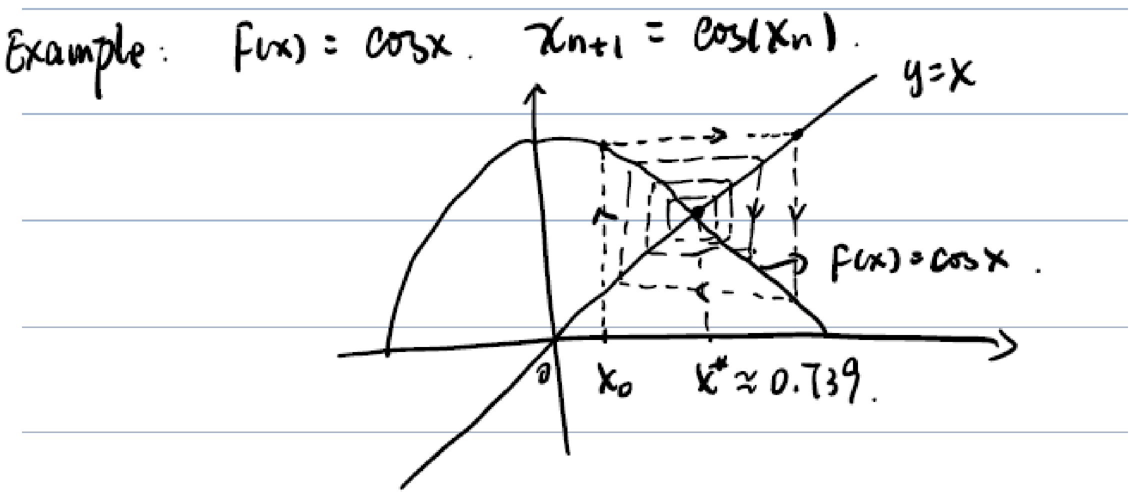

Figure 13: Cobweb and phase portrait for $x_{n+1} = \cos(x_n)$

2.2.3 Linear stability analysis (of fixed point)

Consider $x_{n+1} = F(x_n)$. Then x^* is a fixed point of map if $x^* = F(x^*)$. Consider a nearby orbit $x_n = x^* + y_n$ where $y_n \ll 1$. Does the derivation y_n grow or decay as n increases?. Using linear approximation, we have:

$$x_{n+1} = x^* + y_{n+1} = F(x^* + y_n) = F(x^*) + F'(x^*)y_n + \mathcal{O}(y_n^2)$$

Now since $F(x^*) = x^*$, we have $y_{n+1} = F'(x^*)y_n$. Therefore we get:

$$y_n = (F'(x^*))^n y_0$$

If we define $\lambda = F'(x^*)$ as the **multiplier of** x^*, we have the following cases (the first three cases is called **hyperbolic**):

1. $|\lambda| < 1$: Then x^* is **an attractor (stable)**.
2. $|\lambda| > 1$: Then x^* is **a repeller (unstable)**.
3. $\lambda = 0$, x^* is called **superstable** (because $y_n \sim y_0^{2^n}$)
4. $|\lambda| = 1$. Then the stability cannot be determined by linearization. We need to consider higher order term. This is **non-hyperbolic**.

Example 85. Let $F(x) = \frac{1}{2}(x^3 + x)$. the fixed points are $x^3 = x$ so $x^* = 0, \pm 1$. Now

$$F'(x) = \frac{1}{2}(3x^2 + 1)$$

So

$$F'(0) = \frac{1}{2} < 1$$

and

$$F'(\pm 1) = 2 > 1$$

Therefore 0 is stable and ± 1 are unstable. See (14) for the Cobweb diagram of this.

Example 86. Let $F(\theta) = \theta + \varepsilon \sin(2\theta)$ for $0 < \varepsilon < \frac{1}{2}$. Then the fixed points are when $\sin(2\theta) = 0$, that is, $\theta = 0, \frac{pi}{2}, \pi, \frac{3\pi}{2}$. Then:

$$F'(\theta) = 1 + 2\varepsilon \cos(2\theta)$$

Then:

$$F'(0) = F'(\pi) = 1 + 2\varepsilon > 1: \text{ repelling fixed points}$$
$$F'(\frac{\pi}{2}) = F'(\frac{3\pi}{2}) = 1 - 2\varepsilon < 1: \text{ attracting fixed points}$$

See (15) for Phase Portrait.

Proposition 87. Suppose F is C^1. Let p be a hyperbolic fixed point with $|F'(p)| < 1$. Then there is an open interval U around p such that if $x \in U$, then:

$$\lim_{k \to \infty} F^k(x) = p$$

That is, $U \subseteq W^s(p)$.

Proposition 88. Let p be a hyperbolic fixed point with $|F'(p)| > 1$, then there is an open interval U of p such that, if $x \in U$, $x \neq p$, then there exists $k > 0$ such that $F^k(x) \notin U$.

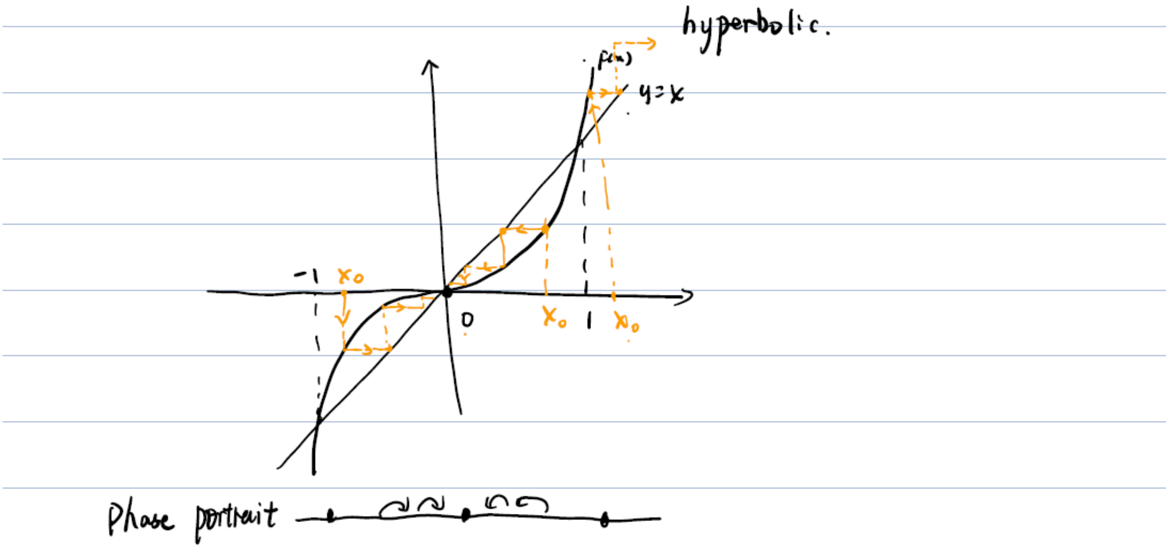

Figure 14: Cobweb and phase portrait for $F(x) = \frac{1}{2}(x^3 + x)$

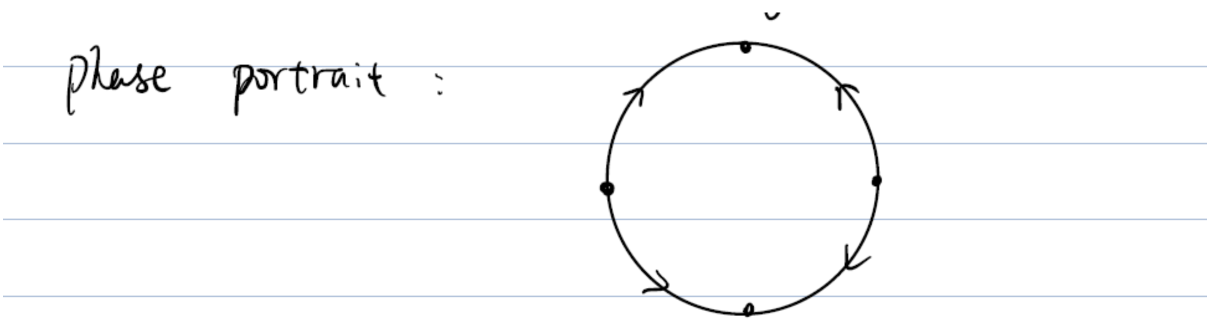

Figure 15: Phase portrait for $F(\theta) = \theta + \varepsilon \sin(2\theta)$

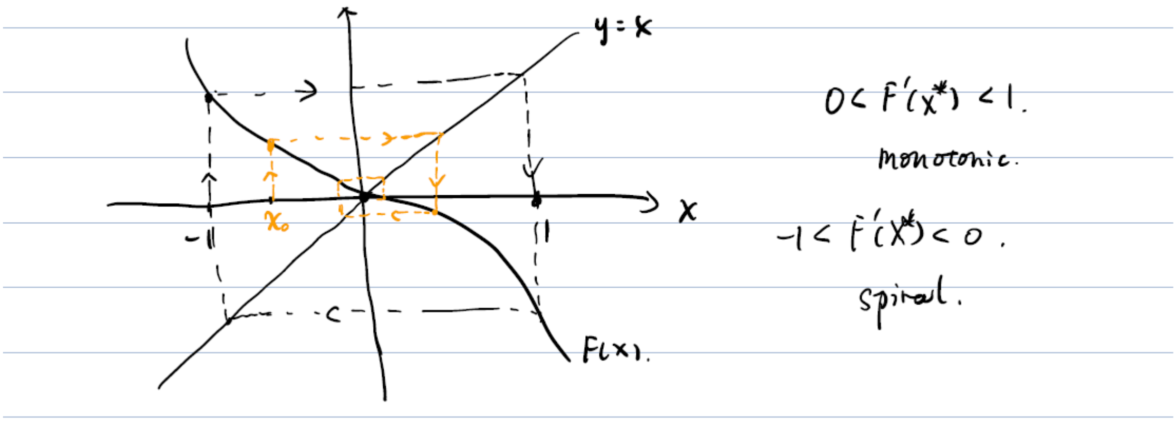

Figure 16: Cobweb for $F(x) = -\frac{1}{2}(x^3 + x)$

2.2.4 Stability of periodic points

Definition 89. Let p be a periodic point of prime period n. (Recall a prime period is the smallest n such that $F^n(p) = p$). The point p is **hyperbolic** if $|(F^n)'(p)| \neq 1$. The number $(F^n)'(p)$ is called the **multiplier of the periodic point**.

1. If $|(F^n)'(p)| < 1$, then p is called **an attractor period point (attractor, sink)**
2. If $|(F^n)'(p)| > 1$, then p is called **a repelling period point (repellor, source)**

Example 90. Consider $F(x) = -\frac{1}{2}(x^3 + x)$. The fixed points are $x^3 = -3x$, so $x^* = 0$. In addition, $F(-1) = 1$ and $F(1) = -1$, therefore ± 1 lie on a periodic orbit of period 2.

1. For the fixed point 0, we get $F'(x) = -\frac{1}{2}(3x^2 + 1)$ so $F'(0) = -\frac{1}{2}$. So 0 is a stable (hyperbolic).
2. For the period-2 points ± 1, we get:

   $$(F^2(\pm 1))' = (F(F(\pm 1)))' = F'(F(\pm 1))F'(\pm 1) = F'(-1)F'(1) = (-2)(-2) = 4 > 1$$

   So ± 1 are periodic points unstable (and hyperbolic).

See (16) for the Cobweb of fixed point x^*. Note that when $0 < F'(x^*) < 1$, it is monotonic. And when $-1 < F'(x^*) < 0$, it is spiral.

Remark 91. Suppose $\{p_1, \ldots, p_N\}$ is a cycle of period n,i.e., $F(p_1) = p_2$, $F(p_2) = p_3, \ldots$, $F(p_n) = p_1$.

1. Then each p_i is a fixed point of the $n-th$ iterate: $F^n(p_i) = p_i$. To analyze the stability of a cycle, we can reduce the problem to a question about the stability of a fixed point. Stability of the cycle $\{p\}_{i=1}^n \Rightarrow$ stability of $\lambda = \frac{d}{dx}(F^n(x))\Big|_{x=p_i}$
2. Note that:

   $$\begin{aligned}\frac{d}{dx}(F^N)(p_i) &= (F^N)'(p_i) = (F(F^{N-1}))'(p_i) \\ &= F'(F^{N-1}(p_i))(F^{N-1})'(p_i) = F'(F^{N-1}(p_i))F'(F^{N-2}(p_i)) \ldots F'(p_i) \\ &= \underbrace{F'(p_N)F'(p_{N-1}) \ldots F'(p_1)}_{N}\end{aligned}$$

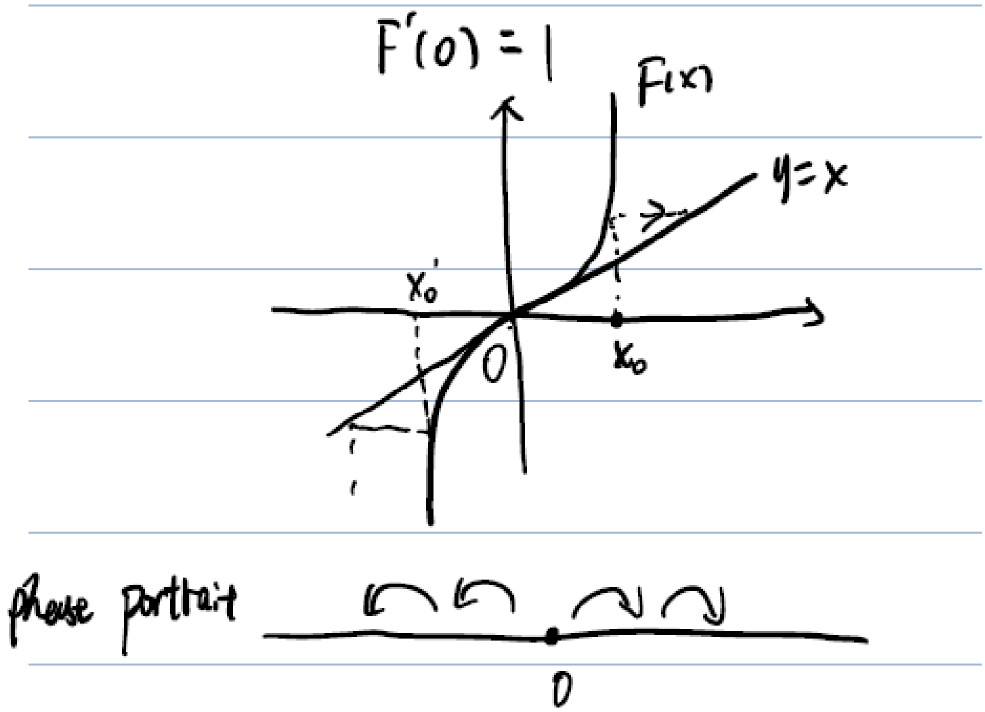

Figure 17: Cobweb for $F(x) = x + x^3$ - repelling non-hyperbolic at 0

We would like to see next some non-hyperbolic points example.

Example 92. Consider the function $F(x) = x + x^3$. Then $F'(x) = 1 + 3x^2$. Therefore at $x = 0$, $F'(0) = 1$ and 0 is a non-hyperbolic point of this function. The Cobweb and phase portrait in figure (17) shows that 0 is a repelling case.

Example 93. Consider $F(x) = x - x^3$. Then $F'(x) = 1 - 3x^2$. 0 is a fixed point with $F(0) = 0$ and $F'(0) = 1$, so 0 is a non-hyperbolic fixed point. The Cobweb diagram and phase portrait in figure (18) show that in this case, 0 is an attractor.

Example 94. Consider $F(x) = x + x^2$. Then $F'(x) = 1 + 2x$. Again, 0 is a fixed point since $F(0) - 0$, but it is non-hyperbolic since $F'(0) = 1$. The Cobweb and Phase Portrait diagram in figure (19) showed this point is "attracting" on the left but "repelling" on the right.

2.3 Day 9

2.3.1 Bifurcation

Definition 95 (bifurcation definition)**.** As we have seen in the Phase portrait, the dynamic of this is very limited, either settle down to equilibrium (stable) or go out to $\pm\infty$. What is interesting about this then? It is the **dependence on parameter**: the qualitative structure of the dynamical system can change as the parameter varies. For instance, fixed points can be created or destroyed, or their stability can be changed. These qualitative change in a dynamic system is called **bifurcation**. The parameter values at which they occur are called **bifurcation points**.

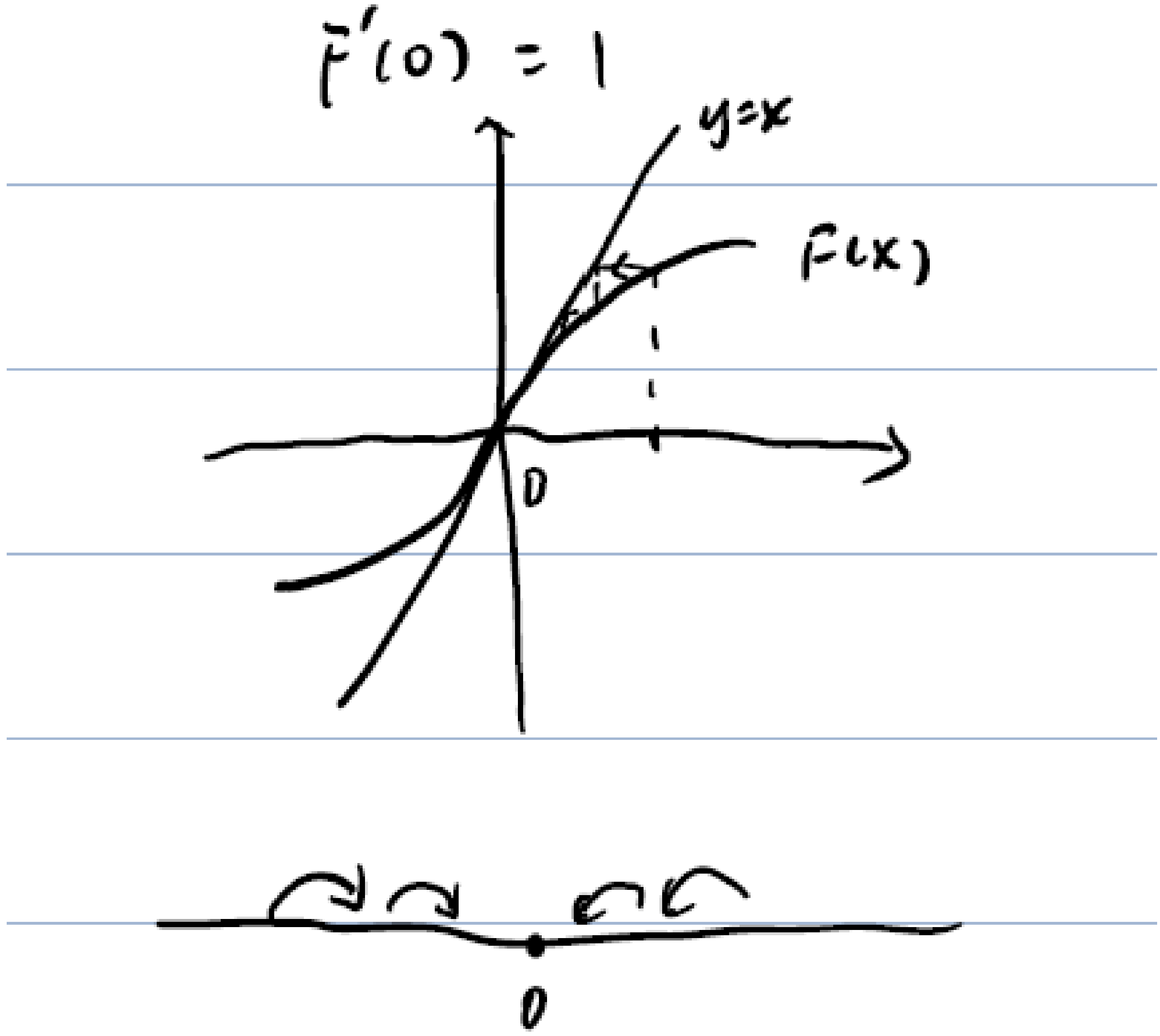

Figure 18: Cobweb for $F(x) = x - x^3$ - attractor non-hyperbolic at 0

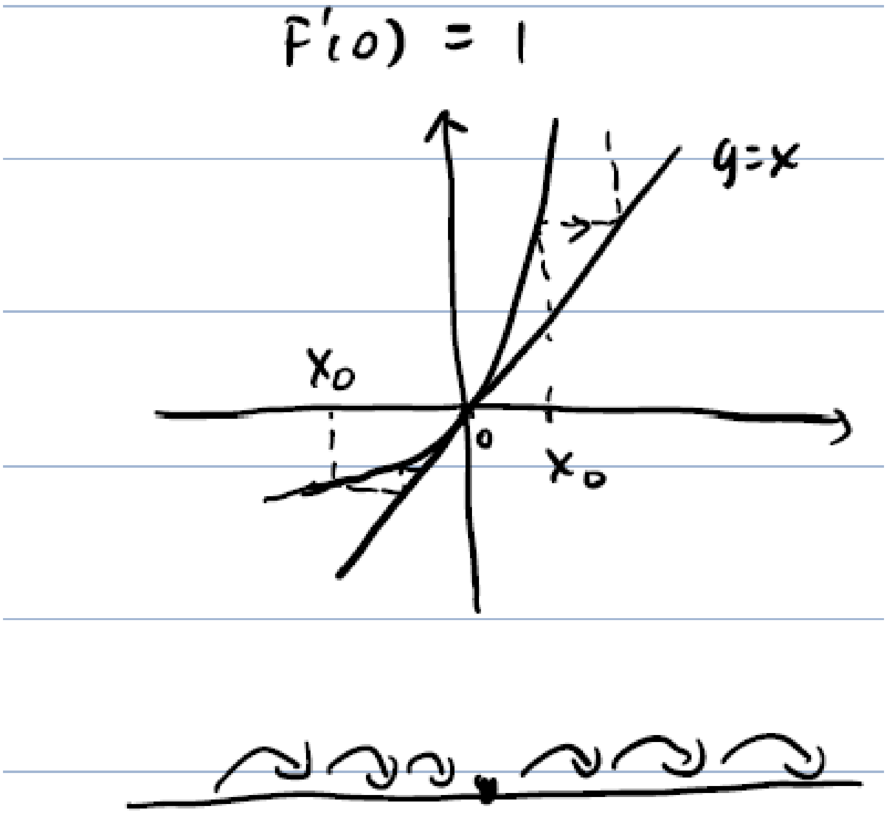

Figure 19: Cobweb for $F(x) = x + x^2$ - both attractor and repellor non-hyperbolic at 0

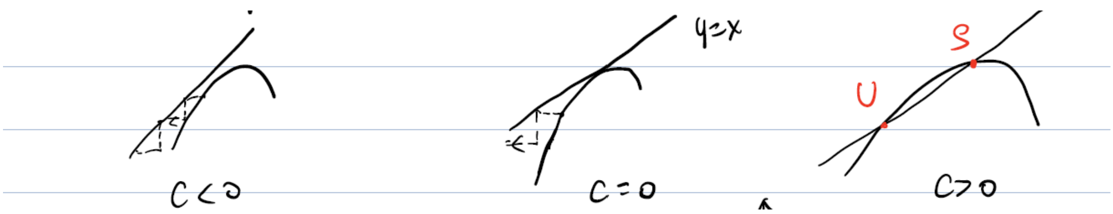

Figure 20: The three cases depends on parameter c for $F(x) = c + x - x^2$

Example 96. Now we introduce parameter c with function $x_{n+1} = F(x_n) = c + x_n - x_n^2$. Their fixed points are $c - x^2 = 0$ so $x = \pm\sqrt{c}$. We have the following cases depend on values c:

1. $c < 0$: no fixed point
2. $c = 0$ unique non-hyperbolic fixed point $x = 0$.
3. $c > 0$: a pair of fixed points, one attracting and one repelling.

We say that the map undergoes a bifurcation as c increases through 0. See figure (20) for the three cases. However, we have another way of describing this situation, using a **bifurcation diagram**: a plot of location of fixed points using parameter value c. In this particular example, we see a situation of a **saddle node bifurcation** (sometimes called tangent bifurcation). See figure (21) for a pictorial.

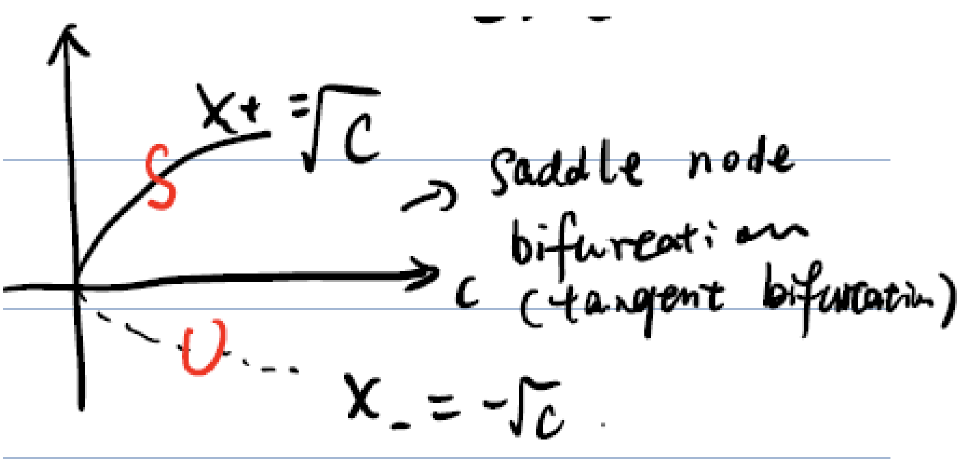

Figure 21: The saddle node bifurcaion parameter c for $F(x) = c + x - x^2$

Definition 97 (saddle node bifurcation)**.** Fixed points can be created or destroyed. **As a parameter varies, the two fixed point move toward each other, collide, and mutually annihilate**. The dash-line usually represent unstable fixed point, while the solid line represent stable fixed point. One of the axis will be the parameter.

Definition 98 (Transcritical bifurcation)**.** There are situations in which a fixed point (or more) must exists for all values of parameters and can never be destroyed. However, such a fixed point may change its stability when parameter varies.

Example 99. Consider $x_{n+1} = F(x_n) = (1+c)x_n - x_n^2$. The fixed points satisfying $x = (1+c)x - x^2$, that is, $x(c-x) = 0$. There are always 2 fixed points: $x^* = 0, c$. Dee figure (22) for graph. The corresponding multiplier is $F'(0) = 1 + c$ and $F'(c) = 1 - c$. Then we have these cases (note that we always have 2 fixed points):

1. When $c > 0$: the $x* = 0$ is unstable (since $F'(0) = 1 + c > 1$ while the $x^* = c$ is stable (since $F'(c) = 1 - c < 1$.

2. when $c < 0$: the $x^* = 0$ is stable (since $F'(0) = 1 + c < 1$ while the $x^* = c$ is unstable (since $F'(c) = 1 - c > 1$.

See figure (23) for a transcritical bifurcation diagram.

Definition 100 (pitchfork bifurcation)**.** This type of bifurcation usually happens in a (physical) system that have a symmetry. There are 2 types: **supercritical pitchfork bifurcation** (think of this like creating more stable) and **subcritical pitchfork bifurcation** (think of this like more unstable).

Example 101. Consider $x_{n+1} = F(x_n) = (1+c)x_n - x_n^3$. Then the fixed points are $x(c - x^2) = 0$. Note that $F'(x) = 1 + c - 3x^2$. Now it depends on the value c that we have 3 cases (see figure (24) for function picture):

1. When $c < 0$, we only have single fixed point $x^* = 0$ and this fixed point is stable.

2. When $c = 0$: also only 1 fixed point, but the fixed point is non-hyperbolic. This case it is tangential.

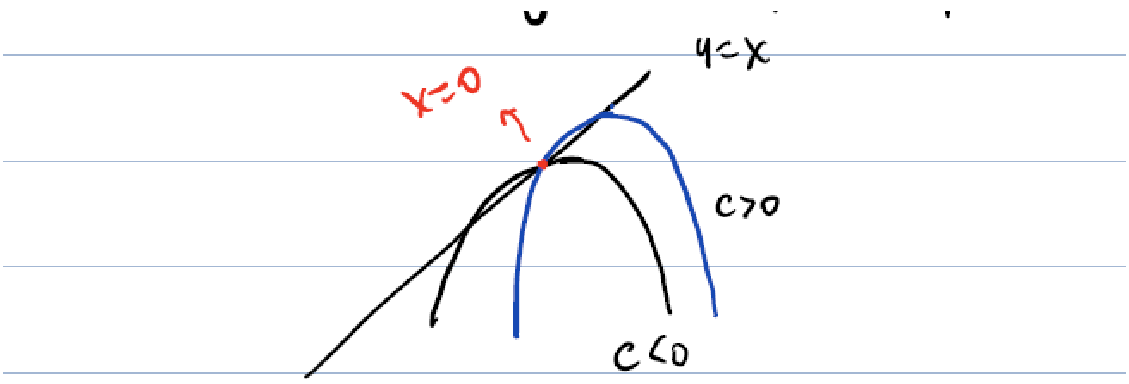

Figure 22: The graph of $F(x)$ which shows 2 fixed points for both cases of $c > 0$ and $c < 0$

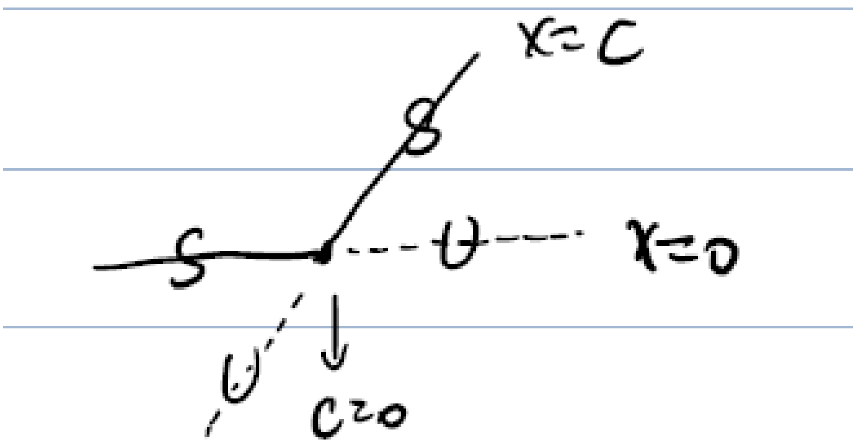

Figure 23: Transcritical bifurcation diagram

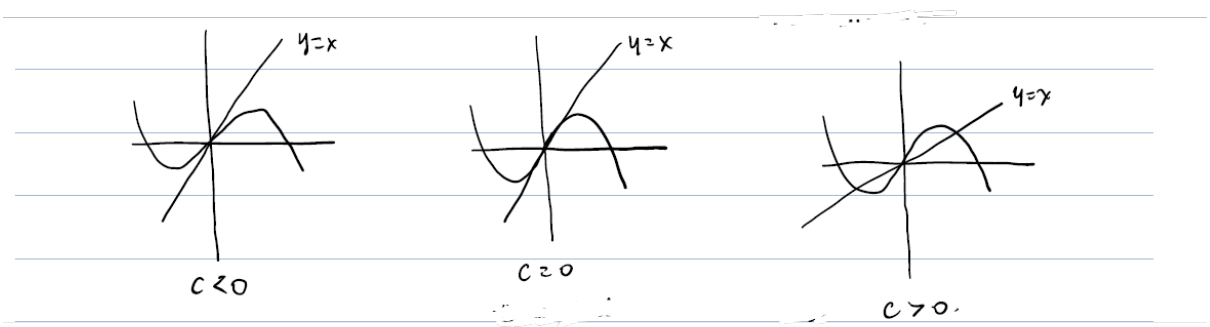

Figure 24: Three cases $c < 0$, $c = 0$ and $c > 0$ of $F(x) = (1 + c)x - x^3$

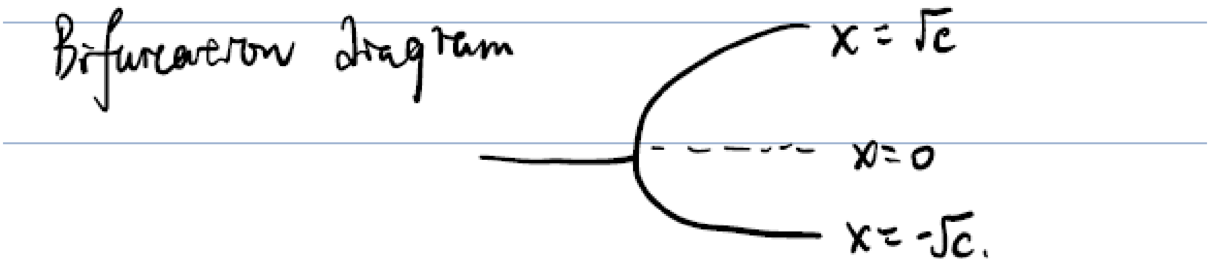

Figure 25: Supercritical pitchfork bifurcation diagram

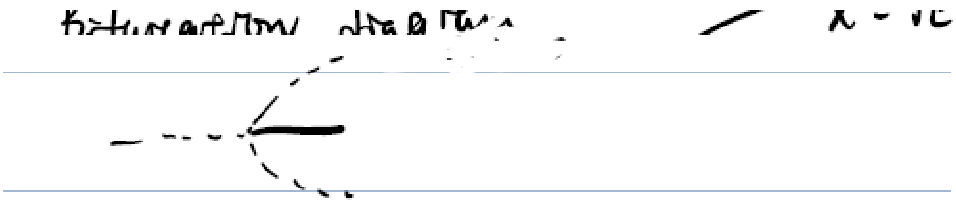

Figure 26: Subcritical pitchfork bifurcation diagram

3. When $c > 0$, we have three fixed points $x^* = 0, \pm\sqrt{c}$. The $x^* = 0$ is unstable while the $\pm\sqrt{c}$ are stable.

And we also have the supercritical pitchfork bifurcation diagram (as we have after $c = 0$, we created more stable fixed points (2) than unstable fixed point (1)). See figure (25) for pictorial.

Example 102. Take $F(x) = (1+c)x + x^3$. In this case we will have an example pictorial of a subcritical pitchfork bifurcation. We have more unstable fixed points than stable fixed points.We started with 1 unstable fixed point, then when $c \geq 0$, we created 2 unstable fixed points and 1 stable fixed point. See figure (26) for pictorial.

2.4 Day 10: Logistic Map analysis

In a paper in 1976, Robert May introduced that even simple non-linear maps could have very complicated dynamic. He gave example of **logistic map**

Definition 103 (Logistic map)**.** The map $F(x) = rx(1-x)$ is called the **logistic map**. We consider $x_{n+1} = F(x_n)$ and will try to find fixed point for this.

Now let us begin the analysis of this map. First note that this map is a parabola with max at $x = \frac{1}{2}$ because:

$$F'(x) = r - 2rx = r(1-2x)$$

In addition, Note that the maximum value is $F(\frac{1}{2}) = \frac{r}{4}$. In addition, if we restrict our domain such that $x \in [0, 1]$, we get that $F(x) = rx(1-x) \geq 0$ given that $r \geq 0$. If we also want $F(x) \leq 1$ for all $x \in [01,]$, we would require the max value of F which is $\frac{r}{4} \leq 1$, so $r \leq 4$. Thus we have $0 \leq r \leq 4$. See figure (27) for picture of logistic map.

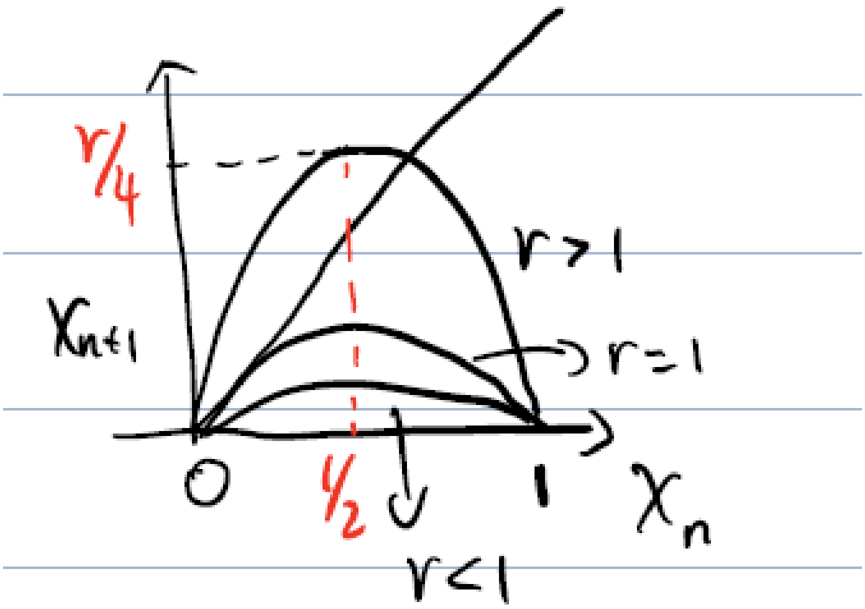

Figure 27: logistic map $F(x) = rx(1-x)$

Now let's analyze the fixed points of $F(x)$. That is, we need:

$$\begin{aligned} rx(1-x) &= x \\ rx^2 - rx + x &= 0 \\ x(rx - r + 1) &= 0 \end{aligned}$$

We got two values:

1. $x_1^* = 0$ will be a fixed point for **all** value r.
2. $x_2^* = 1 - \frac{1}{r}$ is a fixed point only if $r \geq 1$.

Next we consider the linear stability of fixed points:

$$F'(x) = r(1-x) + rx(-1) = r(-2x)$$

Recalled the multiplier $\lambda = |F'(x^*)|$. So we have for each of the fixed point above these cases:

1. For $x_1^* = 0$: $F'(0) = r$. So
   (a) If $0 < r < 1$: 0 is a stable fixed point.
   (b) If $r = 1$: inconclusive
   (c) If $r > 1$: 0 is unstable.
2. For the $x_2^* = 1 - \frac{1}{r}$: we get $F'(x_2^*) = r(1 - 2(1 - \frac{1}{r})) = 2 - r$. So we get these cases:
   (a) If $0 < r < 1$: unstable (but then $x_2^* < 0$, so we really don't have a fixed point in our domain)

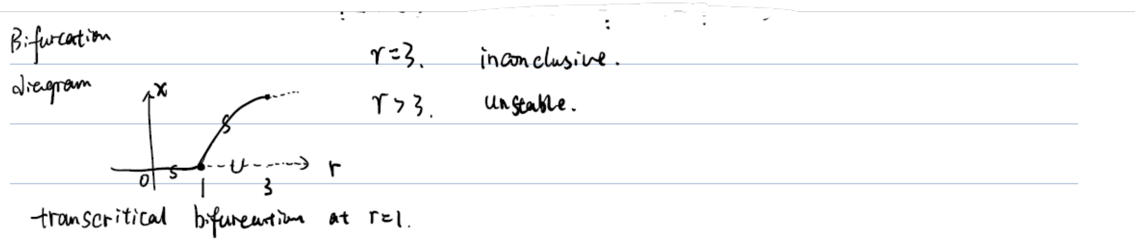

Figure 28: transcritical bifurcation at $r = 1$

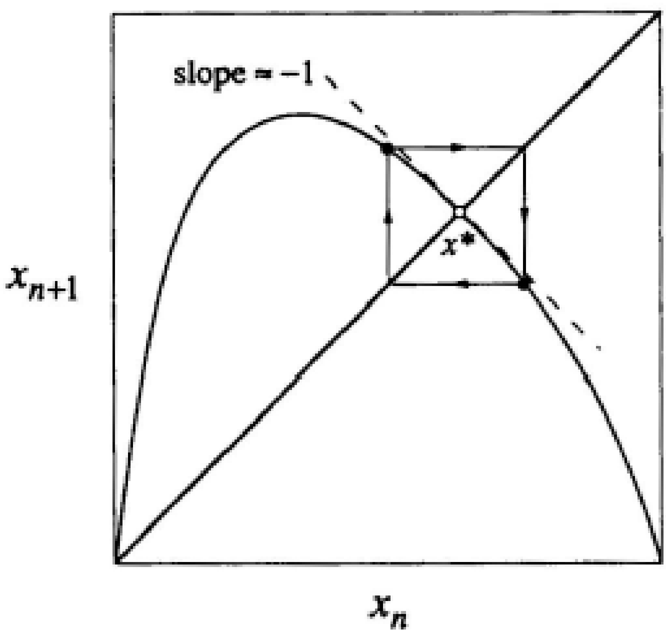

Figure 29: Cobweb shows periodic-2 cycle when $r > 3$

(b) $r = 1$: inconclusive

(c) $1 < r < 3$: stable (since $|F'(x_2^*)| < 1$)

(d) $r = 3$: inconclusive

(e) $r > 3$: unstable

Based on this analysis, we have a situation of transcritical bifurcation diagram at $r = 1$ in figure (28):

Now the interesting cases happen when r passes 3, then $F'(x_2^*)$ will pass through -1. It's interesting from numerical testing value of r we see there is some periodic-2 cycle orbit in the original system (see figure (29) and see that there is a return of x_n that eventually hit the line $y = x$).

So it's natural to look for these period-2 points. That is, we look for p, q such that $F(p) = q$ and $F(q) = p$. In fact, this mean we are looking for fixed point p such that $F^2(p) = p$. see figure (30), we are looking for the two red dots in that figure.

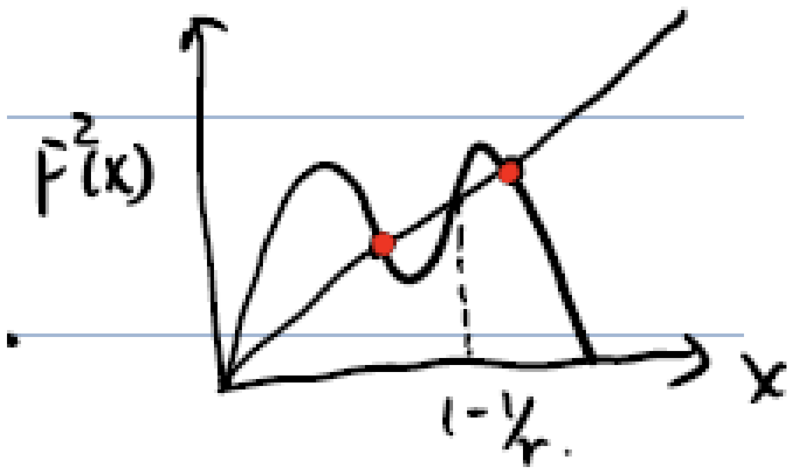

Figure 30: Graph of $F^2(p) = p$ for logistic map

To do this, we need to find $F^2(x)$:

$$\begin{aligned} F(x) &= rx(1-x) \\ F^2(x) = F(F(x)) &= r\left[F(x)(1-F(x))\right] \\ &= r^2x(1-x)(1-rx+rx^2) \end{aligned}$$

So now we set

$$\begin{aligned} F^2(x) &= x \\ r^2x(1-x)(1-rx+rx^2) &= x \end{aligned}$$

Note that $x^* = 0, 1-\frac{1}{r}$ (the fixed points) are both roots of $F^2(x) = x$. To find the other roots, we consider:

$$\begin{aligned} r^2(1-x)(1-rx+rx^2) - 1 &= 0 \\ r^3x^3 - 2r^2x^2 + r^2(r+1)x - r^2 + 1 &= 0 \end{aligned}$$

In order for $x^* = 1-\frac{1}{r}$ to be a root of the above equation, we must have $rx - r + 1$ divide the polynomials. We use long division and get:

$$r^3x^3 - 2r^2x^2 + r^2(r+1)x - r^2 + 1 = (rx - r + 1)(r^2x^2 - (r^2+r)x + (r+1))$$

Therefore our roots will be the root of $r^2x^2 - (r^2+r)x + (r+1)$:

$$x_\pm = \frac{(r^2+r) \pm \sqrt{(r^2+r)^2 - 4r^2(r+1))}}{2r^2} = \frac{r+1 \pm \sqrt{(r-3)(r+1)}}{2r}$$

So our

$$p = \frac{r+1+\sqrt{(r-3)(r+1)}}{2r} \quad q = \frac{r+1-\sqrt{(r-3)(r+1)}}{2r} \text{ exists when } r > 3$$

When $r = 3$:

$$p = q = \frac{r+1}{2r} = \frac{4}{6} = \frac{2}{3} == 1 - \frac{1}{3} = x_2^*$$

Therefore, we get that when $r \to 3^+$, the $p, q \to x_2^*$. And at $r = 3$, a period-2 cycle appears.

Example 104. Verify that $F(p) = q$ and $F(q) = p$ where the p and q are as above.

2.4.1 More on logistic map

Remark 105. The linear stability of the period-2 cycle can help gain more insights into stability of other cycle periodic points.

Remark 106. A strategy that is worth remembering: to analyze the stability of a cycle, we reduce the problem to the question about the stability of a fixed point: The p and q which are the 2-cycle are the fixed points of the **second-iterate map** $F^2(x) = x$.

Recall last time p and q are periodic-2 cycle points of the logistic map, i.e., they are the roots of $F^2(x) = x$. Let's look at their linear stability by looking at the multiplier (recall $F(x) = rx(1-x)$ and $F'(x) = r(1-2x)$):

$$\begin{aligned}\lambda = \frac{d}{dx}(F^2(x))\Big|_{x=p} &= \frac{d}{dx}(F(F(x))\Big|_{x=p} \\ &= F'(F(x))F'(x)\Big|_{x=p} \\ &= F'(q)F'(p) = r(1-2q)r(1-2p) = r^2(1-2(p+q)+4pq)\end{aligned}$$

Since p, q are roots of $r^2x^2 - (r^2+r)x + (r+1) = 0$, we get (compare with the completed factor out $(x-p)(x-q) = x^2 - (p+q)x + pq$):

$$p + q = \frac{r^2+r}{r^2}\ , \text{ and } pq = \frac{r+1}{r^2}$$

Therefore our multiplier is:

$$\lambda = r^2(1 - 2\frac{r^2+r}{r^2} + 4\frac{r+1}{r^2}) = -r^2 + 2r + 4$$

Hence the stability of the 2-cycle is determined by $\lambda = -r^2 + 2r + 4$.

Note that when $r = 3$, the multiplier $\lambda = -3^2 + 2.3 + 4 = 1$ is a possible bifurcation. Because $\frac{d\lambda}{dr} = -2r + 2 < 0$ when $r > 3$, we see that when $r \uparrow$, then $\lambda \downarrow$.

The next possible bifurcation point is at $\lambda = -1$. That is $-r^2 + 2r + 4 = -1$. And we solve this and get $r^2 - 3r - 5 = 0$ so $r = 1 + \sqrt{6}$ (note that we did not take $r = 1 - \sqrt{6} < 0$). Therefore the 2-cycle is (linearly) stable when $3 < r < 1 + \sqrt{6}$ (since the $\lambda \downarrow$, so $\lambda < 1$ when r is in this range). At $r = 1 + \sqrt{6}$, the 2-cycle loses its stability to the creation of a 4-cycle. This is because for $r > 1 + \sqrt{6}$, F^2 has a 2-cycle, therefore F has a 4-cycle. See figure (31) for bifurcation diagram.

Numerical experiments reveal the following periodic-doubling cascade:

1. For $r_1 = 3$ and $r_2 = 1 + \sqrt{6} \approx 3.449$: stable 2-cycle for $r_1 < r < r_2$.
2. $r_3 \approx 3.54409$: Stable 2^2-cycle for $r_2 < r < r_3$
3. $r_4 \approx 3.5644$: Stable 2^3-cycle for $r_3 < r < r_4$
4. $r_5 \approx 3.568759$: Stable 2^4-cycle for $r_4 < r < r_5$
5. ...

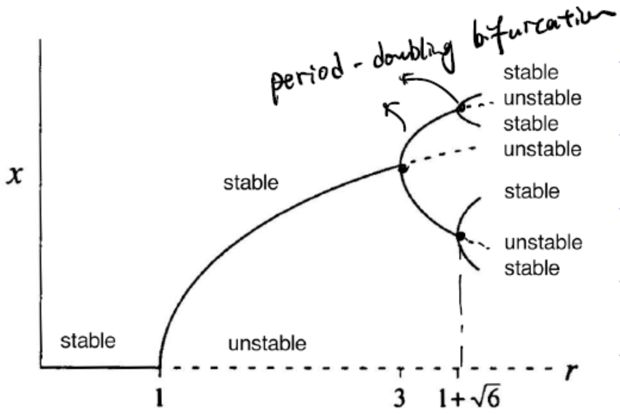

Figure 31: Bifurcation diagram upto 4-cycle points for logistic map

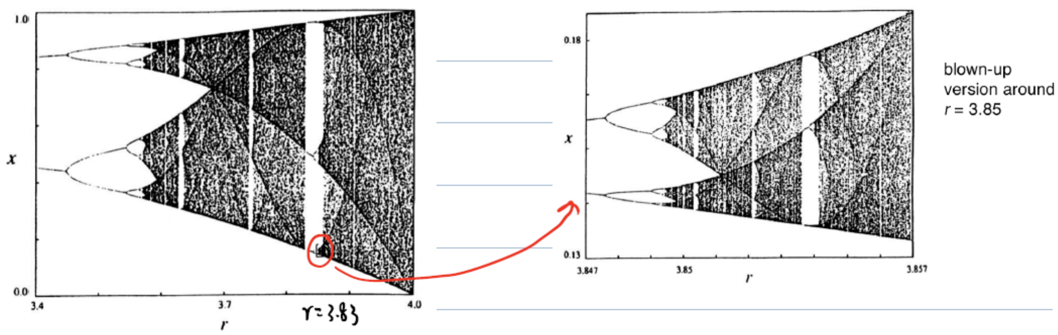

Figure 32: What happens after r_∞

We have stable 2^m cycle for $r_m < r < r_{m+1}$. Note successive bifurcation comes faster and faster and that:

$$\lim_{m\to\infty} r_m = r_\infty \approx 3.569446$$

The distance between successive transitions shrinks by a constant factor called the **Feigenbaum constant**:

$$s = \lim_{m\to\infty} \frac{r + m - r_{m-1}}{r_{m+1} - r_m} \approx 4.669$$

What happens after $r = r_\infty$? See figure (32). This figure has an interesting mixture of order and chaos, with periodic windows interspersed between chaotic clouds of dots. The large window beginning near $r \approx 3.83$ contains a stable period-3 cycle. A blown-up of part of the periodic-3 window is shown in the right part of this figure. Interestingly. we see that a copy of the orbit diagram reappear in miniature.

2.4.2 Lyapunov exponent

Definition 107 (chaotic)**.** To be called **chaotic**, a system should show sensitivity dependence on initial conditions, in the sense that neighboring orbits separate exponentially fast.

Let us consider two nearby initial points x_0 and $x_0 + S_0$, initial separation S_0 is extremely small. We called S_n separation after n iterations.

Definition 108 (Lyapunov exponent)**.** If $|S_n| \approx |S_0|e^{\lambda n}$, then λ is called the **Lyapunov exponent**. In addition, if $\lambda > 0$, this is a sign of chaos.

Remark 109. We will now begin to derive the formula for Lyapunov exponent λ_l.

Consider $x_{n+1} = F(x_n)$. Consider 2 nearby points x_0 and $\hat{x_0}$, with $S_0 = \hat{x_0} - x_0$ and $|S_0|$ is small. The two corresponding orbits are: $\{x_n\}$ and $\{\hat{x_n}\}$. We define:

$$\begin{aligned} S_n &= \hat{x_n} - x_n = F^n(\hat{x_0}) - F^n(x_0) = F^n(x_0 + S_0) - F^n(x_0) \text{ , since } |S_0| \ll 1 \\ &\approx \frac{d}{dx} F^n(x)\bigg|_{x=x_0} S_0 \end{aligned}$$

By using the chain rule, we get:

$$\frac{d}{dx} F^n(x)\bigg|_{x=x_0} = F'(x_{n-1})F'(x_{n-2}) \dots F'(x_0)$$

Therefore

$$\left|\frac{S_n}{S_0}\right| = \prod_{i=0}^{n} |F'(x_i)|$$

Now taking natural log give:

$$\frac{1}{n} \ln\left|\frac{S_n}{S_0}\right| = \frac{1}{n}\left(\ln \prod_{i=0}^{n} |F'(x_i)|\right) = \frac{1}{n}\sum_{i=0}^{n} \ln|F'(x_i)|$$

So the Lyapunov exponent for the orbit starting at x_0 is:

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \ln\left|\frac{S_n}{S_0}\right| = \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n} \ln|F'(x_i)|, \text{ if the limit exists}$$

Remark 110. We observe the following:

1. λ depends on x_0
2. For stable fixed points and cycles, λ is negative
3. for chaos, λ is positive.

Example 111. Suppose that F has a stable p-cycle containting the point x_0. Show that the Lyapunov exponent $\lambda < 0$. If the cycle is superstable, show that $\lambda = -\infty$. We showed it as follow:
Since x_0 is in the $p-$ cycle, so x_0 is a fixed point of F^p. The cycle is stable so $|(F^p)'(x_0)| < 1$. Then taking natural log we get $\ln|(F^p)'(x_0)| < \ln 1 = 0$. Then for p-cycle:

$$\lambda = \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \ln|F'(x_i)| = \frac{1}{p}\sum_{i=0}^{p-1} \ln|F'(x_i)|$$

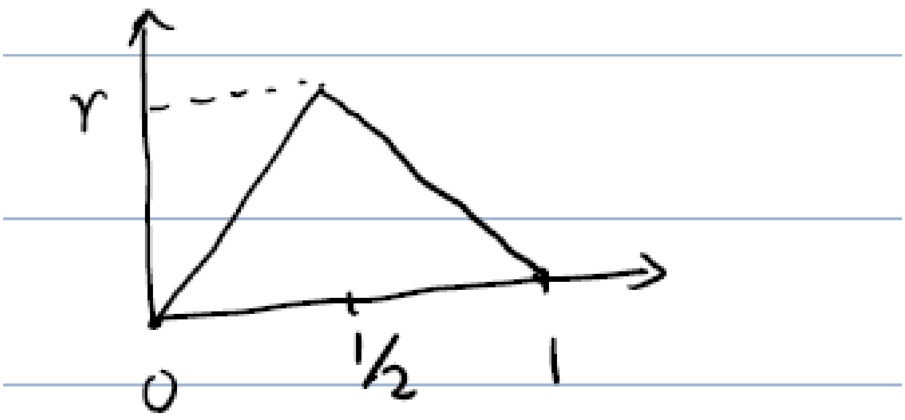

Figure 33: Tent Map

Now note that:

$$\frac{1}{p}\sum_{i=0}^{p-1}\ln|F'(x_i)| = \frac{1}{p}\ln(\prod_{i=0}^{p-1}|F'(x_i)|) = \frac{1}{p}\ln|(F^p)(x_0| < 0$$

So $\lambda < 0$.
If the cycle is superstable (recall a fixed point x^* is superstable if $|F'(x^*)| = 0$), so $|\frac{d}{dx}F^p(x_0)| = 0$. So $\lambda = \frac{1}{p}\ln(0) = -\infty$.

Example 112 (Tent map)**.** Find the Lyapunov exponent for the tent map $x_{n+1} = F(x_n)$ where $(r > 0)$:

$$F(x) = \begin{cases} 2rx & \text{if } 0 \leq x \leq \frac{1}{2} \\ 2r(1-x) & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases}$$

See the figure (33) for Tent map picture. We get the derivative:

$$F'(x) = \begin{cases} 2r & \text{if } 0 \leq x \leq \frac{1}{2} \\ -2r & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases}$$

For a typical orbit $\{x_0, x_1, \ldots\}$, we calculate:

$$\lambda = \lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\ln|F'(x_i)| = \lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\ln(2r) = \ln(2r)$$

Thus we have the cases:

1. If $2r < 1$ or $r < \frac{1}{2}$, then $\lambda_l < 0$ so there is no sensitive dependence on initial condition.

2. if $2r > 1$, or $r > \frac{1}{2}$, then $\lambda_l > 0$ so sensitive dependence on initial condition (which mean possible chaos).

There are the Cobweb diagram when $r = \frac{3}{4}$ in figure (34).

$r = 3/4.$

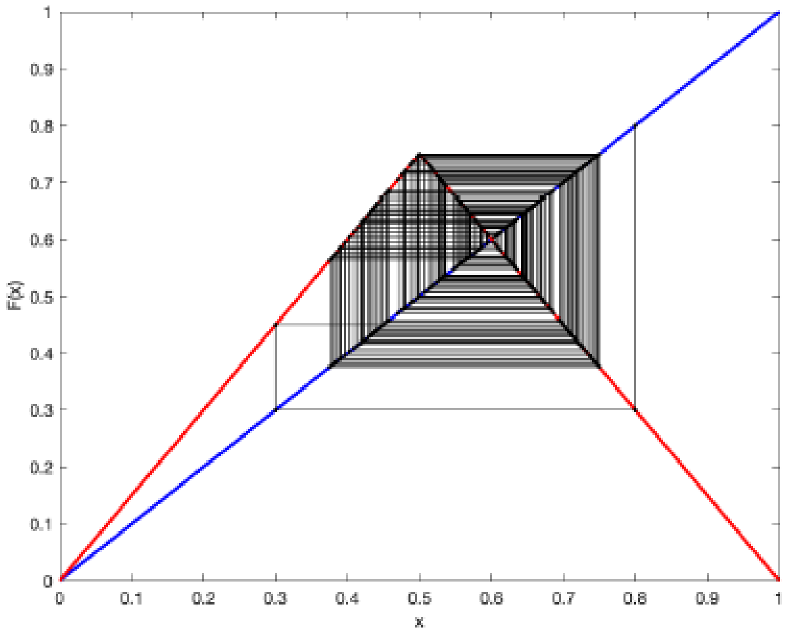

Figure 34: Cobweb for tent map with $r = \frac{3}{4}$

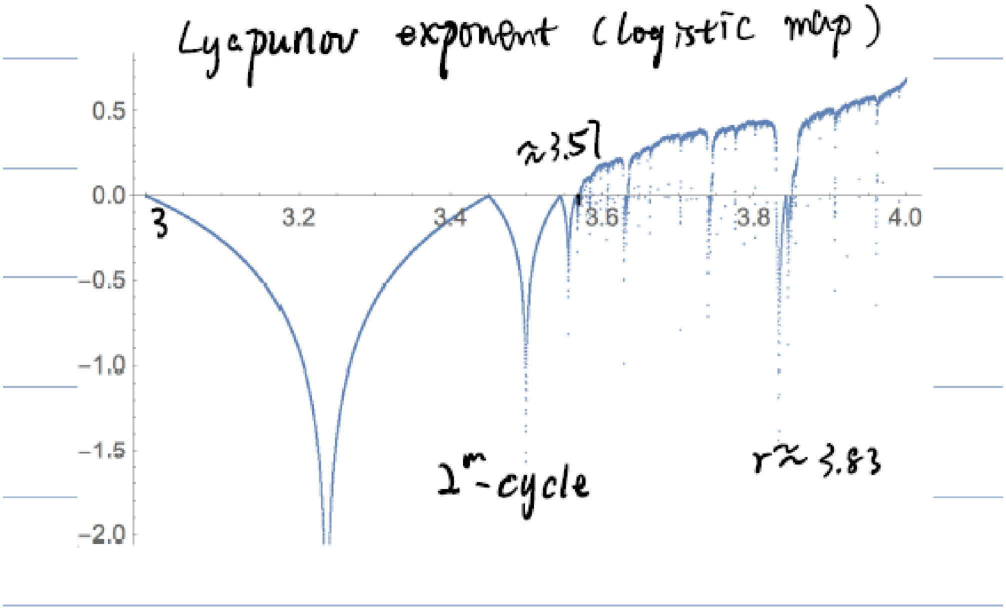

Figure 35: Lyapunov exponent for logistic map

Example 113. Find the Lyapunov exponent for the logistic map numerically.
See figure (35) for the diagram. To do this numerically, first we fix some value of r. Then starting from a random initial condition, iterate the map long enough to allow transients to decay, say 300 iterates or so. Then we compute a large number of additional iterates, say $10,000$. You only need to store the current value of x_n, not all the previous iterates. Then you compute $\ln(F'(x_n)) = \ln|r - 2rx_n|$ and add it to the sum of the previous logarithm. The Liapunov exponent is then obtained by dividing the total by $10,000$. Repeat this procedure for the next random r. The λ remains negative for $r < r_\infty \approx 3.57$, and get to zero at the periodic-doubling bifurcation. Note that the negative spikes correspond to the 2^n-cycle. The onset of chaos is visible near $r \approx 3.57$ where the λ first become positive. For $r > 3.57$, note that the Liapunov exponent generally increases, except for the dips caused by windows of periodic. The large dip is due to period-3 windows near 3.83. In fact, from previous example, we should get at all the dips, $\lambda = -\infty$, because superstable is guaranteed to occur somewhere near the middle of each dip.

2.5 Day 11:

2.5.1 Topology conjugacy

Definition 114 (Topological conjugacy)**.** Let $f : I \to I$ and $g : J \to J$ be two maps. f and g are said to be topological conjugacy if there exists a homeomorphism $h : I \to J$ such that $h \circ f = g \circ h$. The homeomorphism h is called the topological conjugacy.

Remark 115. Maps that are topologically conjugagte are completely equivament in terms of their dynamics. This is because if $h \circ f = g \circ h$, and let p be a fixed point of f, so $f(p) = p$, then $h(p)$ is a fixed point of $g(x)$: $g(h(p)) = h(f(p)) = h(p)$. Therefore the map $h : \text{Per}_n(f) \to \text{Per}_n(g)$ is a $1-1$ correspondence.

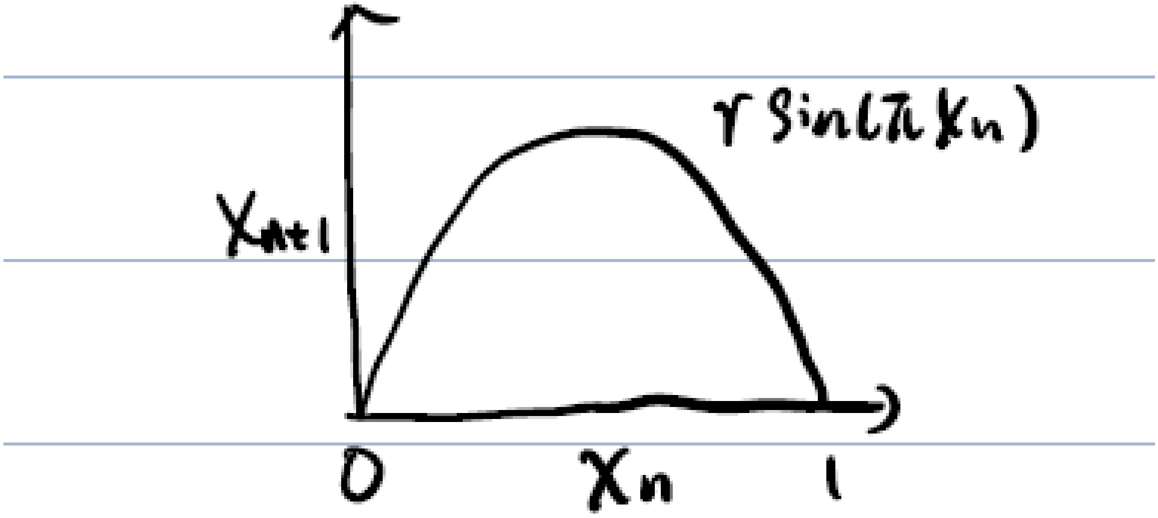

Figure 36: Unimodal sine function $r\sin(\pi x)$

Example 116. Show that the tent map with $r = 1$ and the logistic map with $r = 4$ are conjugate to each other. Recall the tent map when $r = 1$ is

$$T(x) = \begin{cases} 2x & \text{if } 0 \le x \le \frac{1}{2} \\ 2(1-x) & \text{if } \frac{1}{2} \le x \le 1 \end{cases}$$

And the logistic map when $r = 4$ is $F(x) = 4x(1-x)$. The conjugacy function is $h(x) = \frac{1}{2}(1 - \cos \pi x)$. Note that $h : [0,1] \to [0,1]$ and is a homeomorphism. Then for $x \in [0, \frac{1}{2}$ we have:

$$\begin{aligned} h(T(x)) &= \frac{1}{2}(1 - \cos(\pi 2x)) = \sin^2(\pi x) = 1 - \cos^2(\pi x) \\ &= (1 - \cos(\pi x))(1 + \cos(\pi x)) \\ &= 4(\frac{1}{2} - \frac{1}{2}\cos(\pi x))(\frac{1}{2} + \frac{1}{2}\cos(\pi x)) \\ &= 4h(x)(1 - h(x)) = F(h(x)) \end{aligned}$$

2.5.2 Universality

This idea is best illustrative via example. Consider the trigonometric function $x_{n+1} = r\sin(\pi x_n)$ for $r \in [0,1]$ and $x \in [0,1]$. The illustrative picture is in figure (36). It has the same shape as the logistic map, and these shape are called **unimodal**. It is suprising that the dynamical of this map and the logistic map are identical.

Recall the logistic map $F = 4x(1-x)$. The fixed points are $4x(1-x) = x$ so $x = 0, \frac{3}{4}$. The periodic-2 cycle are $F(F(x)) = 0$ and earlier we solve and get 2^2 fixed points of this, with the extra are $\frac{5 \pm \sqrt{5}}{5}$. For the period-3 cycle, we also get $2^3 = 8$ fixed points: 2 fixed points of $F(x)$, and 2 periodic 2 cycle, and 2 period-3 cycle (see figure (37)). In general, F^k has 2^k fixed points in $[0,1]$. For every positive integer k, there is an orbit of period-k.

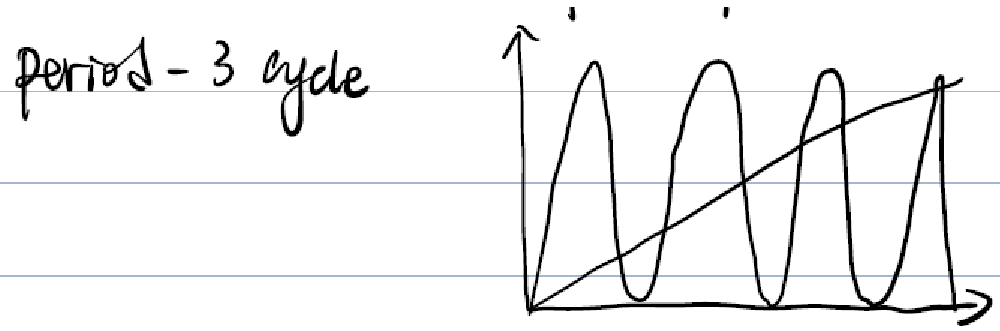

Figure 37: period-3 cycle for logistic map $4x(1-x)$

2.5.3 Itinerary

For the logistic map with $x \in [0,1]$, consider $L = [0, \frac{1}{2}]$ and $R = [\frac{1}{2}, 1]$.Consider an intial point x_0. Construct a trajectory/orbit starting from x_0, and we can also construct the itinerary by listing the symbols, corresponding to each element in the orbit:

$$\begin{array}{cccccc} \text{orbit} & x_0 & x_1 & x_2 & x_3 & \dots \\ \text{itinerary} & L & R & L & R & R & \dots \end{array}$$

Example 117. For $x_0 = \frac{1}{4}$ and $x_{n+1} = 4x_n(1-x_n)$:
The orbit is $\{\frac{1}{4}, \frac{3}{4}, \frac{3}{4}, \dots\}$
The itinerary is $\{L, R, R, \dots\} = L\overline{R}$.
When $x_0 = \frac{1}{3}$ we have:
The orbit is $\{\frac{1}{3}, \frac{8}{9}, \frac{32}{81}, \dots\}$
The itinerary is $\{L, R, L, \dots\}$.
Special case when $x_0 = \frac{1}{2}$ we have the orbit is $\{\frac{1}{2}, 1, 0, 0, 0, 0, \dots\}$ and the itinerary is $RR\overline{L}$ or $LR\overline{L}$, since the L and R overlap at $\frac{1}{2}$. With the exception of this case, the itinerary is always uniquely defined.

Question 118. What is the set of points whose itinerary begins with LR, RL?

Example 119. When $x_0 = \frac{1}{4}$ then it lies in LRR.
When $x_0 = \frac{1}{3}$ then it lies in LRL.
We can count the number of R in the sequence:
odd: ....R, ...L
even: ...L, ...R

The image of each interval L and R contains both L and R, so the transition map (see figure (39)) is fully connected (every symbol is connected to every other symbol by an arrow). So all possible sequence of L and R are possible.
We use itinerary to understand the sensitivity dependence on initial condition (for example, in the case logistic map with $r = 4$).
In specifying the first k symbols of the itinerary, we have 2^k choices. We have $2 * k$ subintervals. It can be proved that each subinterval has size less than $\frac{\pi}{2^{k+1}}$.
Consider an interval I corresponding to $S_1 S_2 \dots S_k$ for $S_i \in \{L, R\}$. I is an interval of length $< \frac{\pi}{2^{k+1}}$. These are four subintervals of I:

1. $S_1 \dots S_k LL$

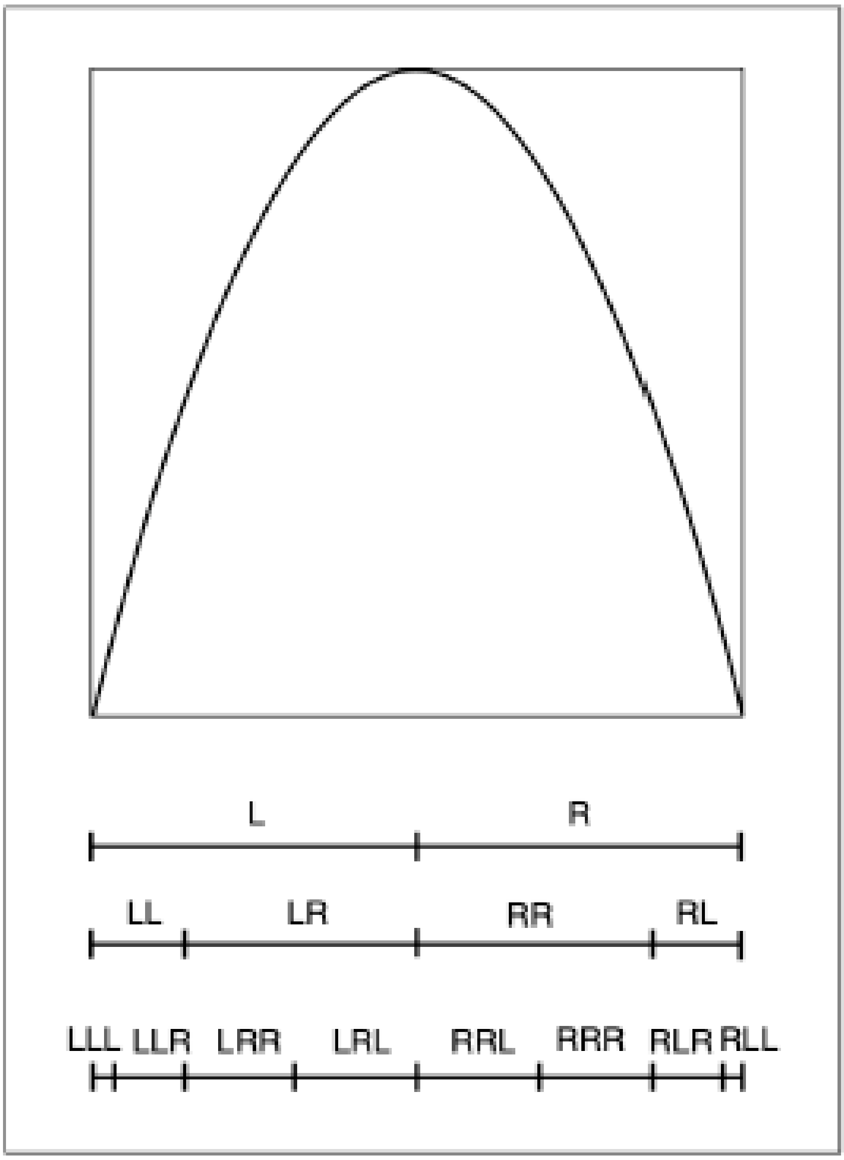

Figure 38: Itinerary counting

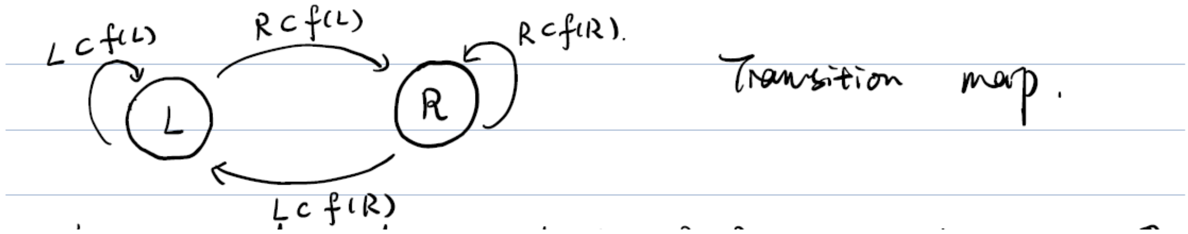

Figure 39: transient map

2. $S_1 \dots S_k LR$
3. $S_1 \dots S_k RL$
4. $S_1 \dots S_k RR$

This is because when we choose one point from each subinterval, 4 initial conditions, distance $< \frac{\pi}{2^{k+1}}$. They are mapped to LL,LR,RL,RR, at least $\frac{1}{4}$ apart. So the sensitive dependence on initial condition.

2.6 Day 12: Stability of higher dimension

2.6.1 Maps of higher dimension

Consider $\overrightarrow{x_{n+1}} = \overrightarrow{F}(\overrightarrow{x_n})$: m-dimensional case. In which we have:

$$\overrightarrow{x_{n+1}} = (x_{1n}, x_{2n}, \dots, x_{mn})^T$$

and

$$\overrightarrow{F} = (F_1, F_2, \dots, F_m)^T$$

The fixed point is

$$\overrightarrow{F}(\overrightarrow{x^*}) = \overrightarrow{x^*}$$

Example 120. Consider $\overrightarrow{F} : \mathbb{R}^3 \to \mathbb{R}^2$ by:

$$\overrightarrow{F} = \begin{pmatrix} \frac{1}{2}x \\ 2y - \frac{15}{8}x^3 \end{pmatrix}$$

where:

$$\overrightarrow{x} = \begin{pmatrix} x \\ y \end{pmatrix}$$

The fixed points $\overrightarrow{F}(\overrightarrow{x}) = \overrightarrow{x}$ so:

$$\frac{1}{2}x = x$$
$$2y - \frac{15}{8}x^3 = y$$

This give $\overrightarrow{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is a (only) fixed point. That is $\overrightarrow{F}(\overrightarrow{0}) = \overrightarrow{0}$.

Example 121 (Horseshoe map)**.** See picture figure (40).

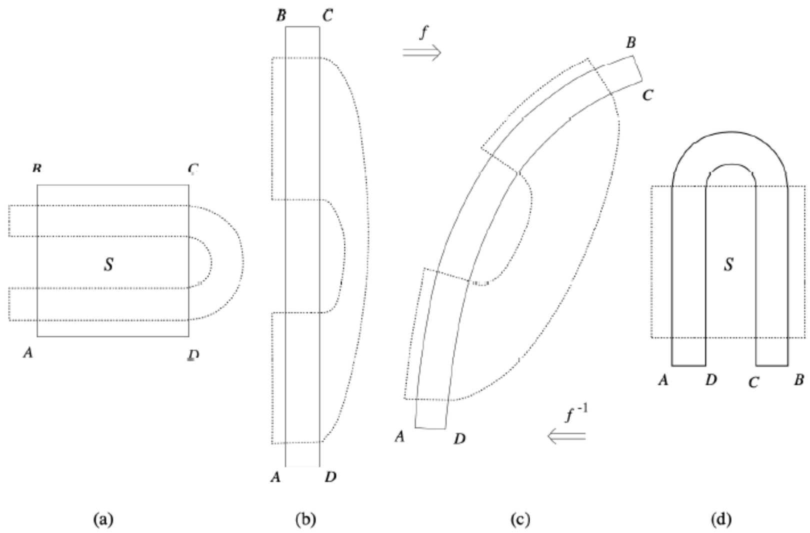

FIGURE 1.6. Construction of the horseshoe map.

Figure 40: horseshoe map

Example 122 (Henon map)**.** Consider

$$T(\overrightarrow{x}) = T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a - x^2 + by \\ x \end{pmatrix}$$

We denote $x_{n+1} = a - x_n^2 + by_n$ and $y_{n+1} = x_n$. What is the fixed point of this map? We set:

$$\begin{aligned} a - x^2 + by &= x \\ x &= y \end{aligned}$$

And solve we get:

$$\begin{aligned} a - x^2 + bx &= x \\ x^2 + (1-b)x - a &= 0 \\ x &= \frac{-(1-b) \pm \sqrt{(1-b)^2 + 4a}}{2} \end{aligned}$$

And the fixed points exists only if $(1-b)^2 + 4a \geq 0$.

Question 123. How to find fixed points of the n-th iterate? $\overrightarrow{F}^n(\overrightarrow{x}) = \overrightarrow{x}$?

Example 124. Henon map, and we are looking for period-2 points:

$$\begin{cases} x_1 &= a - x_0^2 + by_0 \\ y_1 &= x_0 \end{cases}$$

And:

$$\begin{cases} x_2 &= a - x_1^2 + by_1 \\ y_2 &= x_1 \end{cases}$$

Setting $x_2 = x_0$ and $y_2 = y_0$ for period-2 cycle points.

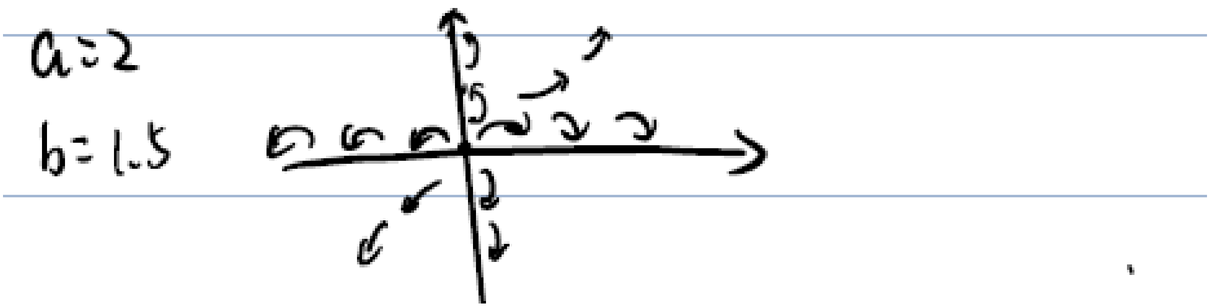

Figure 41: When $|a|, |b| > 1$

2.6.2 Stability of Linear map

Consider the linear maps $F(\overrightarrow{x}) = A\overrightarrow{x}$:

$$F(\overrightarrow{x}) = A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_{11}x + a_{12}y \\ a_{21}x + a_{22}y \end{pmatrix}$$

Every linear map has a fixed point at the origin. Let's take a look at the stability of the origin.

Example 125. Consider

$$A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

The eigenvalues are a, b assuming $a \neq b$. And the eigenvector are $(1.0)^T$ and $(0, 1)^T$. In addition, note that:

$$A^n = \begin{pmatrix} a^n & 0 \\ 0 & b^n \end{pmatrix}$$

It is decoupled (a system is called decoupled if the variables do not interact with each other, each variable can be solved for independently, without anything about the other)

$$\begin{cases} x_{n+1} & = a^n x_1 \\ y_{n+1} & = b^n y_1 \end{cases}$$

We observe the following cases:

1. When $|a|, |b| > 1$. We pick a value for $a = 2$ and $b = 1.5$. Observe the $x_{n+1} = a^n x_1$ (where $x_1 = 1$) and $y_{n+1} = b^n y_1$ (with $y_1 = 0$), then do the same but with $x_1 = 0$ and $y_1 = 0$. Also test these when $x_1 = 1, y_1 = 1$ and when $x_1 - -1, y_1 = -1$. See figure (41) for the phase portrait in 2-D.

2. When $|a|, |b| < 1$. For instance, take $a = \frac{1}{2}$ and $b = \frac{1}{3}$. We note that when initial points on x, y axis, it stay on x, y axis. See figure (42) for picture.

3. When $|a| > 1 > |b|$ or $|b| > 1 > |a|$. In one direction it expands, and in one direction it contracts. See figure (43) for the case $a = 2, b = \frac{1}{2}$ and see figure (44) for the case $a = 2, b = -\frac{1}{2}$.

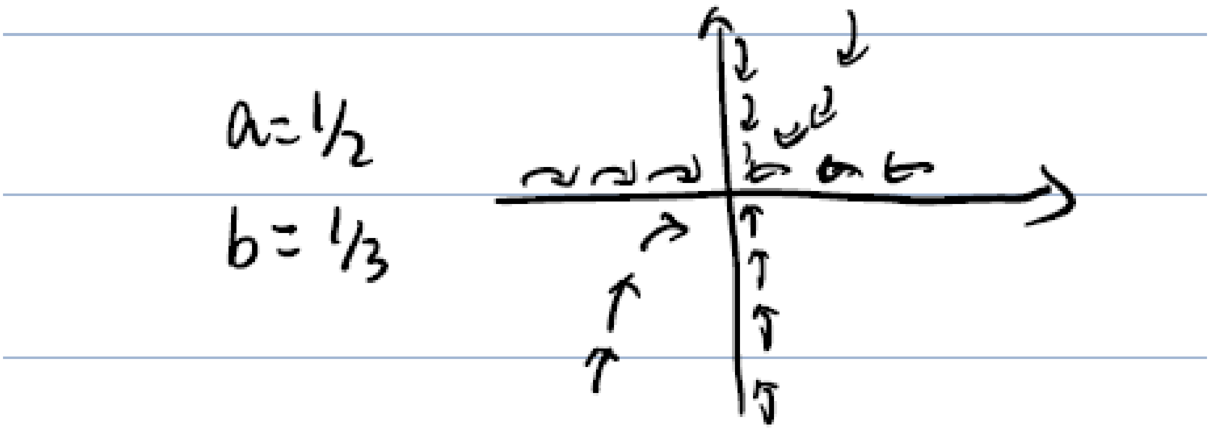

Figure 42: When $|a|, |b| < 1$

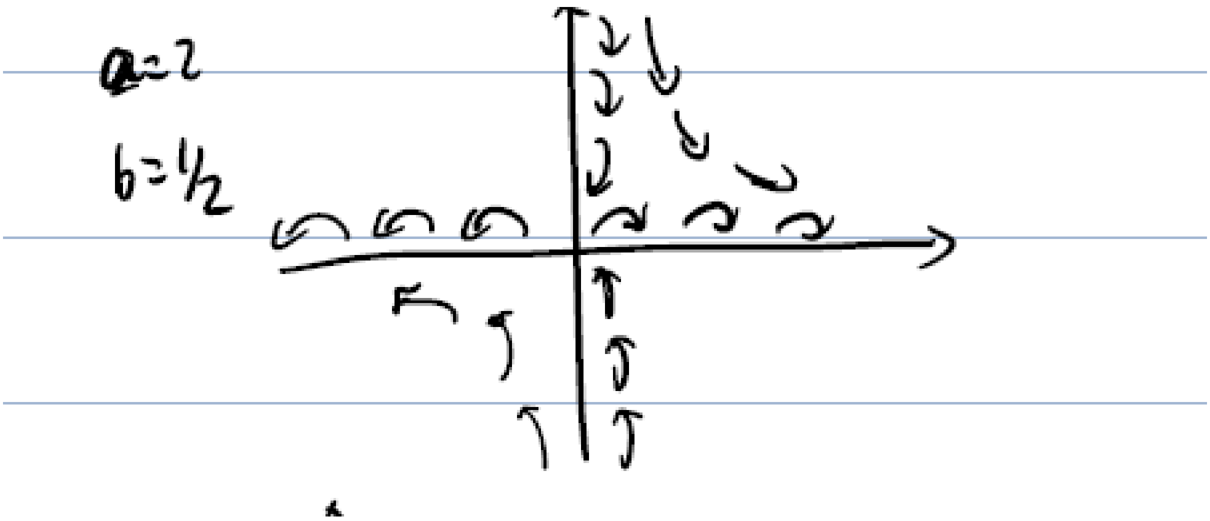

Figure 43: When $|a| < 1 < |b|$

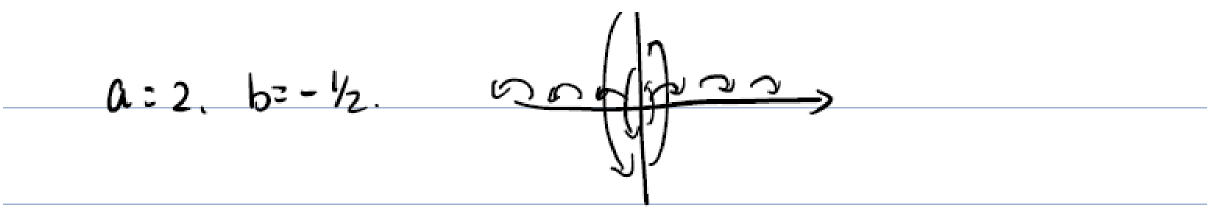

Figure 44: When $|b| < 1 < |a|$

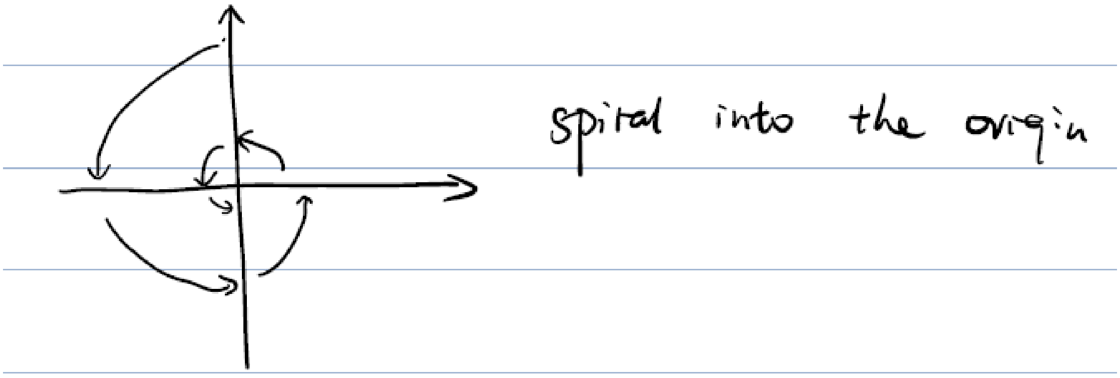

Figure 45: Spiral into the origin

Example 126. Consider
$$A = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$$
Then the eigenvalues is a. Then:
$$A^n \begin{pmatrix} x \\ y \end{pmatrix} = a^{n-1} \begin{pmatrix} a & n \\ 0 & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = a^{n-1} \begin{pmatrix} ax + ny \\ ay \end{pmatrix}$$
will **sink** when $|a| < 1$ and will be a **source** when $|a| > 1$.

Example 127. Consider
$$A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$
The eigenvalues are $a \pm bi$ and the eigenvector are $(\pm i, 1)$. Change of variable $r = \sqrt{a^2 + b^2}$ with $a = r\cos\theta$ and $b = r\sin\theta$ and $\theta = \arctan(\frac{b}{a})$, then our map is:
$$A = r \begin{pmatrix} \frac{a}{r} & -\frac{b}{r} \\ \frac{b}{r} & \frac{a}{r} \end{pmatrix} = \sqrt{a^2 + b^2} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$
Ax rotates the point $\overrightarrow{x}$ about the origin by angle θ (rotation) and multiplies the distance by $\sqrt{a^2 + b^2}$ (dilation). It will be **sink** when $\sqrt{a^2 + b^2} < 1$ and a **source** when $\sqrt{a^2 + b^2} > 1$. An example when $a = 0, b = \frac{1}{2}$ so $\theta = \frac{\pi}{2}$ will give the phase portrait as spiral into the origin (see figure (45)):

2.6.3 Stability of Non-linear map

For non-linear map, the fixed points are not necessarily at $(0, 0)$ origin. The stability of the fixed points depends on the **linearization near fixed point**.
In higher dimensions, the linearization depends on the Jacobian matrix D:
$$\overrightarrow{F}(\overrightarrow{x^*} + \overrightarrow{y}) - \overrightarrow{F}(\overrightarrow{x^*}) \approx D\overrightarrow{F}(\overrightarrow{x^*}).\overrightarrow{y^*}$$

Definition 128 (hyperbolic fixed point)**.** A fixed point $\overrightarrow{p}$ for $\overrightarrow{F} : \mathbb{R}^n \to \mathbb{R}^n$ is called **hyperbolic** if $D\overrightarrow{F}(p)$ has no eigenvalues on the unit circle, where $D\overrightarrow{F}(\overrightarrow{p})$ is the Jacobian matrix at $\overrightarrow{p}$. If $\overrightarrow{p}$ is periodic of period n, then $\overrightarrow{p}$ is **hyperbolic** (periodic) if $D\overrightarrow{F}^n(\overrightarrow{p})$ has no eigenvalue on the unit circle.

Remark 129. We obersve the following:

1. For periodic points, the eigenvalues of the Jacobian matrix $D\overrightarrow{F}^n$ are the same for each point on the orbit.

2. By Chain rule we get:

$$\begin{aligned} D\overrightarrow{F}^n(\overrightarrow{x}) &= D\overrightarrow{F}(\overrightarrow{F}^{n-1}(\overrightarrow{x})).D\overrightarrow{F}^{n-1}(\overrightarrow{x}) \\ &= D\overrightarrow{F}(\overrightarrow{F}^{n-1}(\overrightarrow{x})).D\overrightarrow{F}(\overrightarrow{F}^{n-2}(\overrightarrow{x}))\dots D\overrightarrow{F}(\overrightarrow{x}) \end{aligned}$$

Definition 130 (sink-source-saddle point)**.** Let $\overrightarrow{F}^n(\overrightarrow{p}) = \overrightarrow{p}$. Then:

1. $\overrightarrow{p}$ is a **sink** (or **attracting periodic point**) if all of the eigenvalues of $D\overrightarrow{F}^n(\overrightarrow{p})$ are less than one in absolute value.

2. $\overrightarrow{p}$ is a **source** (or **repelling periodic point**) if all of the eigenvalues of $D\overrightarrow{F}^n(\overrightarrow{p})$ are greater than one in absolute value.

3. $\overrightarrow{p}$ is a **saddle point** if some of the eigenvalues of $D\overrightarrow{F}^n(\overrightarrow{p})$ are larger and some are less than one in absolute value.

Theorem 131 (stable set/manifold and unstable set/manifold)**.** *We have these two cases:*

1. *Suppose F has an attracting fixed point at p. Then there is an open set about p in which all points tend to p under forward iteration of F. The largest such open set is called the stable set or* ***basin of attraction*** *of p, denoted $W^s(p)$.*

2. *Suppose F has a repelling fixed point at p. Then there is an open set about p in which all points tend to p under backward iteration of F. The largest such open set is called the stable set, denoted $W^u(p)$.*

3. *saddle points? unknown*

Example 132. Consider

$$\overrightarrow{F}(\overrightarrow{x}) = \begin{pmatrix} 2x \\ \frac{y}{2} \end{pmatrix}$$

the fixed point is

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

And the Jacobian matrix is:

$$DF = \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

It has two eigen value $\lambda_1 = 2$ and $\lambda_2 = \frac{1}{2}$. So $(0,0)^T$ is a saddle point. We get the eigenvectors as follow:

1. For $\lambda_1 = 2$ the eigenvector is

$$\overrightarrow{v_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \textbf{ unstable manifold}$$

2. For $\lambda_1 = \frac{1}{2}$ the eigenvector is

$$\overrightarrow{v_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \textbf{stable manifold}$$

Definition 133 (stable/unstable manifold)**.** Recall the following:

1. The **stable manifold** of p is $\{\overrightarrow{x} : |\overrightarrow{F}^n(\overrightarrow{x}) - \overrightarrow{p}| \to 0 \text{ as } n \to \infty\}$
2. The **unstable manifold** of p is $\{\overrightarrow{x} : |\overrightarrow{F}^{-n}(\overrightarrow{x}) - \overrightarrow{p}| \to 0 \text{ as } n \to \infty\}$

Note that these are invariant sets.

Example 134. Consider

$$F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{2}x \\ 2y - \frac{15}{8}x^3 \end{pmatrix}$$

Note that $\overrightarrow{F}(\overrightarrow{0}) = \overrightarrow{0}$ and the Jacobian matrix is

$$DF = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{bmatrix}$$

So $\overrightarrow{0}$ is a saddle. We have the following:

1. The unstable manifold: the y-axis since:

$$F\begin{pmatrix} 0 \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 2y \end{pmatrix}$$

2. The stable manifold: $y = x^3$ since:

$$F\begin{pmatrix} x \\ x^3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}x \\ (\frac{1}{2}x)^3 \end{pmatrix}$$

F is topologically conjugate to

$$L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{2}x \\ 2y \end{pmatrix}$$

via

$$h\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ x^3 - y \end{pmatrix}$$

Verify that $F \circ h = h \circ L$. So h maps the stable/unstable subspaces from L onto the stable/unstable manifold of F.

Theorem 135 (stable and unstable manifold Theorem)**.** *Suppose f has a fixed point at p. Let v^s be an eigenvector corresponding to s where $|s| < 1$. And let v^u be an eigenvector corresponding to u with $|u| > 1$. Then both the stable manifold W^s and the unstable manifold W^u are one dimensional manifolds that contain p. The vector v^s is tangent to W^s at p, and v^u is tangent to W^u at p.*

Example 136. The Henon map:

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a - x^2 + by \\ x \end{pmatrix}$$

For the following values of a and b:

1. $a = 0$ and $b = 0.4$ we get the fixed points $(0, 0)$ and $(-0.6), (-0.6)$.
2. for $a = 0.43$ and $b = 4$, we get periodic-2 points

2.6.4 Bifurcation of higher dimension

Example 137. Consider

$$\begin{cases} x_{n+1} &= F_1(x_n) = r + x_n - x_n^2 \\ y_{n+1} &= F_2(y_n) = \frac{1}{2}y_n \end{cases}$$

This will give saddle node bifurcation.

Example 138. Consider

$$\begin{cases} x_{n+1} &= e^{x_n} - \lambda \\ y_{n+1} &= -\frac{\lambda}{2}\arctan(y_n) \end{cases}$$

We calculate $F_1' = e^x$ and $F_2' = -\frac{\lambda}{2}\frac{1}{1y^2}$, therefore:

1. When $0 < \lambda < 1$: no fixed point
2. Saddle node bifurcation at $\lambda = 1$
3. $1 < \lambda < 2$: 2 fixed points: 1 attracting and 1 saddle
4. When $\lambda = 2$: period-doubling bifurcation
5. $\lambda > 2$:?

Example 139. Consider

$$F\begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

We have the following cases:

1. If $\alpha \neq 0$ and $\lambda < 1$, then 0 is an attracting fixed point. All nearby points spiral toward the origin
2. If $\alpha \neq 0$ and $\lambda > 1$, then 0 is a repellor
3. If $\lambda = 1$: we only have rotation, no expansion or contraction.

We see the case of **Hopf bifurcation**: fixed points become stable or unstable, unstable or stable, circle appears.

3 Continuous dynamical systems

3.1 Day 13: continuous dynamical system

3.1.1 General theory for continuous dynamical system

Consider $\dot{x} = f(x)$: one-dimensional or first order system.
If $f = f(t, x)$, we define $x_0 = t$ and $x_1 = x$, the original system can be written as:

$$\frac{d}{dt}\begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = \begin{pmatrix} 1 \\ f(x_0, x_1) \end{pmatrix}$$

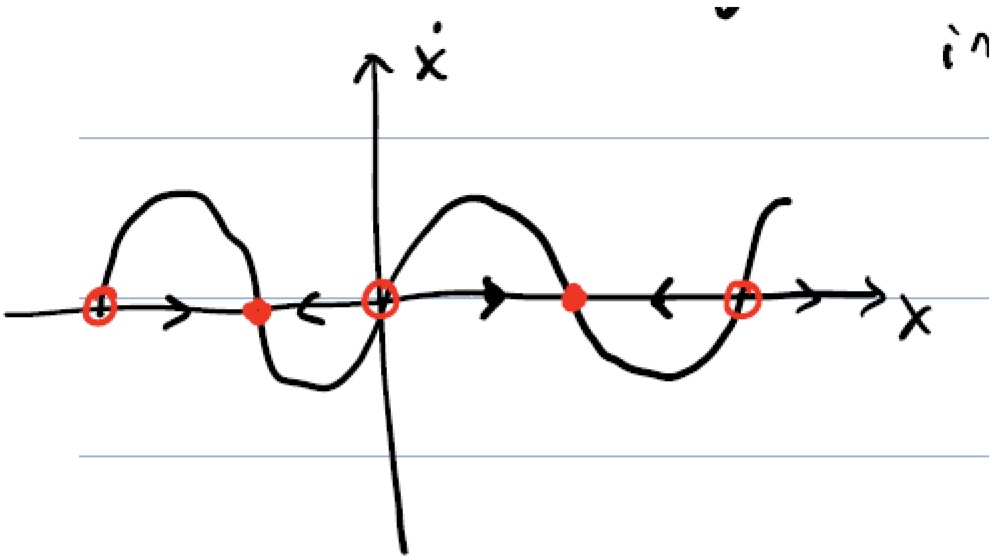

Figure 46: Graphical analysis for $\sin x$

Question 140. For a given initial point x_0, what is the behavior of $x(t)$ as $t \to \infty$?

Example 141. We consider the system $\dot{x} = \sin x$ where $\dot{x} = \frac{dx}{dt}$. Then we get:

$$\frac{dx}{dt} = \sin x$$
$$\frac{dx}{\sin x} = dt$$
$$t = \int \csc x dx = -\ln|\csc x + \cot x| + C$$

If $t = 0$, and $x = x_0$, we get

$$t = \ln|\frac{\csc x_0 + \cot x_0}{\csc x + \cot x}|$$

This is hard to interpret!!

This is why we would like to do a different analysis, the **graphical analysis** in the next section.

3.1.2 Graphical analysis

Assume t represents time, x is the position of an imaginary particle moving along the x-axis. The $\dot{x}$ is the velocity. So take an example with $\dot{x} = \sin x$, this give the vector field on the line. Moreover, $\dot{x} = 0$ is when the particle stays at x_0. The points satisfying $\dot{x} = 0$ are called fixed points.

1. $x = 2k\pi$ for $k = 0, \pm1, \pm2, \ldots$: repellers/sources: unstable fixed points.
2. $x = 2k\pi + \pi$ for $k = 0, \pm1, \pm2, \ldots$ attractors/sinks: stable fixed points.

See figure (46) for graphical visualization.

For any given initial condition, check $\dot{x}$ and get the following cases:

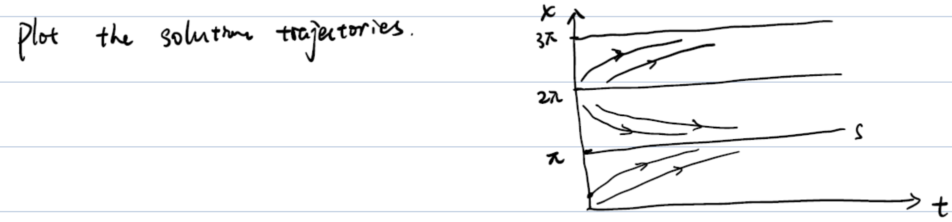

Figure 47: Solution trajectories for $\sin x$

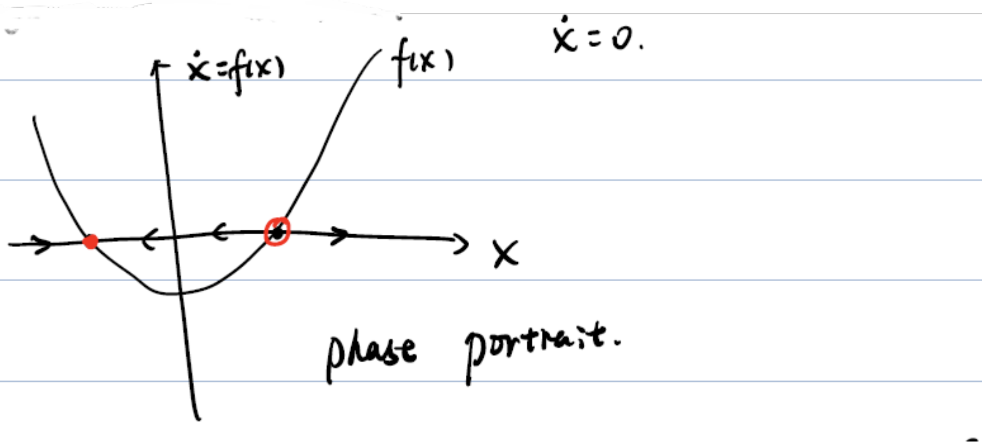

Figure 48: General graphical analysis $\dot{x} = f(x)$

1. If $\dot{x} > 0$, the solution heads to the right and approaches the nearest stable fixed. point.
2. If $\dot{x} < 0$, the solution heads to the left and approaches the nearest stable fixed point.
3. If $\dot{x} = 0$, x remain constant.

We have the solution trajectories in figure (47).

For a general one-dimensional system $\dot{x} = f(x)$:
To analyze the behavior of the solution for a given initial condition, we need to draw the graph of $f(x)$, and sketch the vector field on the real line. See figure (48) for visualization.

Example 142. Consider $\dot{x} = x^2 - 1$. The function is $f(x) = x^2 - 1$. The fixed points are $f(x) = 0$ which gives $x = \pm 1$. The $x = -1$ it is stable (locally stable) and at $x = 1$, it is unstable. See figure (49) for visualization.

Example 143. Consider $\dot{x} = a - bx$ for $a, b > 0$. The fixed point is $x^* = \frac{a}{b}$ is stable (globally stable). See figure(50) for visualization.

3.1.3 linear stability analysis for continuous dynamical system

Consider $\dot{x} = f(x)$ and x^* satisfies $f(x^*) = 0$ is a fixed point. Introduce a small pertubation $\eta(t)$ away from x^*: $x(t) = x^* + \eta(t)$:

$$\dot{x} = \dot{\eta} = f(x^* + \eta) = f(x^*) + \eta f'(x^*) + O(\eta^2)$$

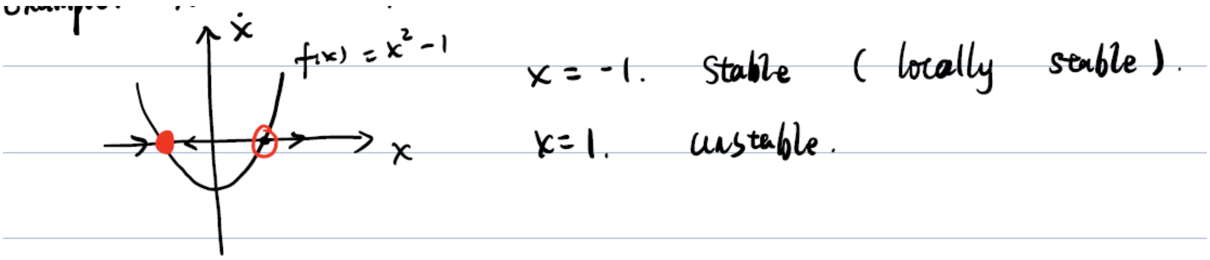

Figure 49: General graphical analysis $\dot{x} = x^2 - 1$

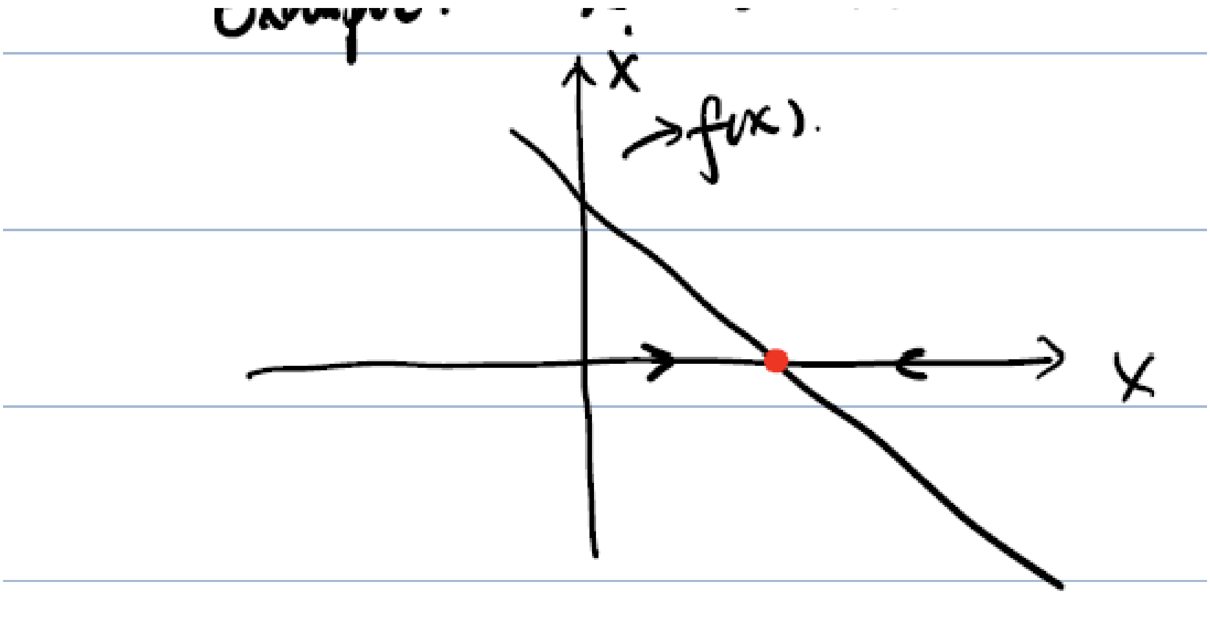

Figure 50: Graphical analysis $\dot{x} = a - bx$

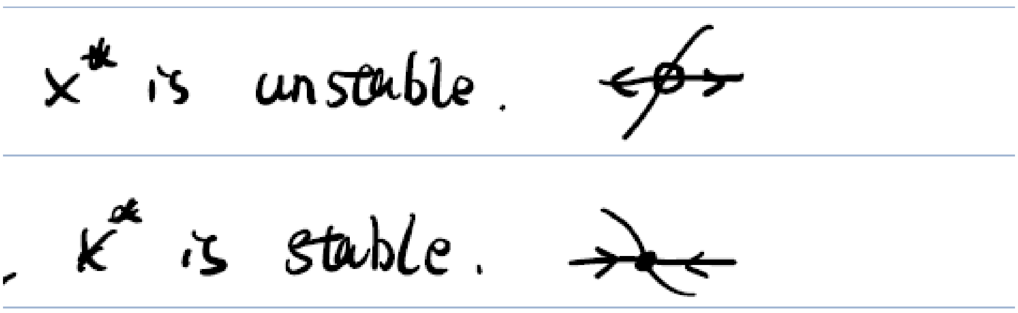

Figure 51: stable vs unstable drawing (filled dot and empty dot)

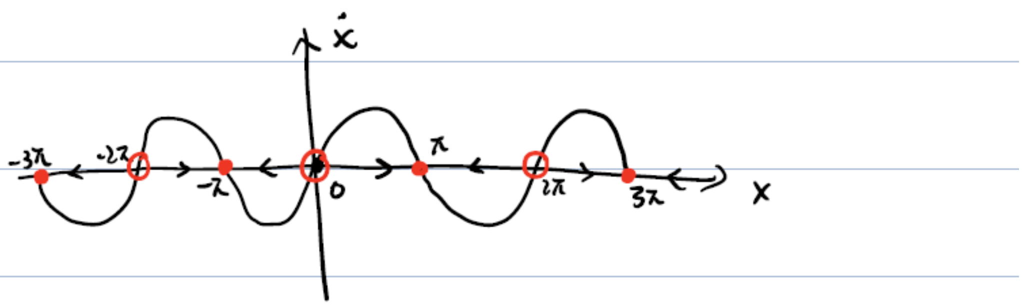

Figure 52: Linear analysis $\dot{x} = \sin x$

So

$$\dot{\eta} \approx \eta f'(x^*) \text{ linearization about } x^*$$

Solving this ODE gives:

$$\eta(x) = \eta_0 e^{f'(x^*)t}$$

where we called $f'(x^*)$ the exponential growth on decay rate.

1. If $f'(x^*) > 0$, then $\eta(t)$ grows exponentially: x^* is unstable.
2. If $f'(x^*) < 0$, $\eta(t)$ decays exponentially: x^* is stable.
3. $f'(x^*) = 0$, $O(\eta^2)$ is not negligible.

See figure (51) for visualization.
The term $\frac{1}{f'(x^*)}$ is a characteristic time scale.

Example 144. $\dot{x} = \sin x$. The fixed point is $\sin x = 0$ solve and give $x^* = k\pi$ for $k = 0, \pm 1, \ldots$. We also have the derivative $f'(x) = \cos x$ so:

$$f'(x^*) = \cos k\pi = \begin{cases} 1 & k \text{ even : unstable} \\ -1 & k \text{ odd : stable} \end{cases}$$

See figure (52) for visualization.

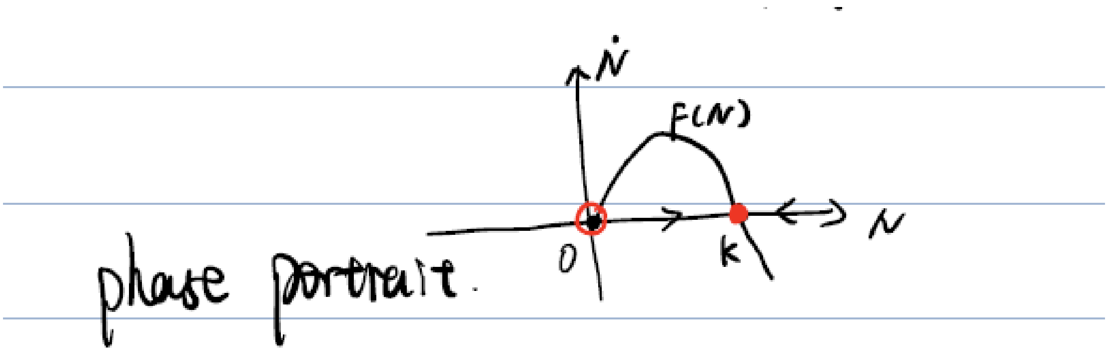

Figure 53: Phase portrait for $\dot{N} = rN(1 - \frac{N}{k})$

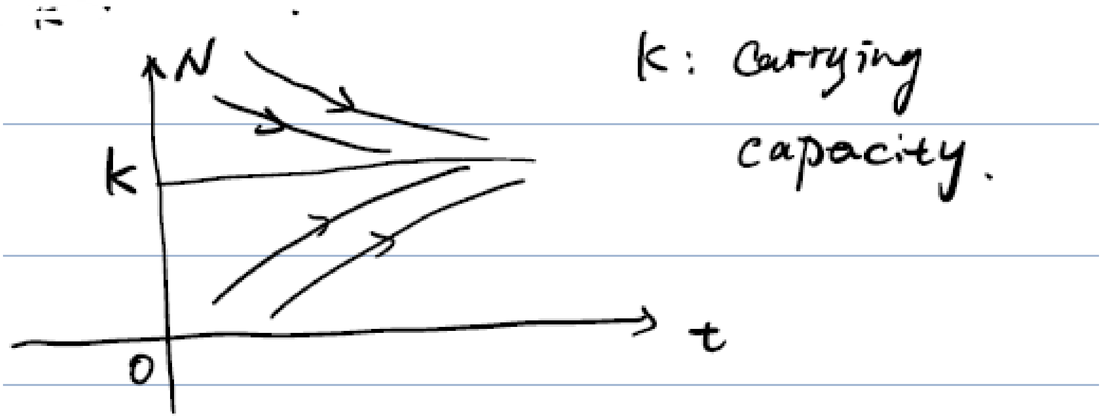

Figure 54: Linear analysis solution trajectory for $\dot{N} = rN(1 - \frac{N}{k})$, k is carrying capacity

Example 145. Consider $\dot{N} = rN(1 - \frac{N}{k})$ for $r > 0$, $k > 0$. Then the fixed points are:

$$rN(1 - \frac{N}{k}) = 0$$

So $N^* = 0$ and $N^* = k$. In addition, the derivative of $f(N) = rN(1-\frac{N}{k})$ is $f'(N) = r(1-\frac{2N}{k})$. Therefore see figure (53) for phase portrait and figure (54) for solution trajectory:

1. When $N^* = 0$, $f'(0) = r$ so 0 is unstable.
2. When $N^* = k$, $f'(k) = r(1 - \frac{2k}{k}) = -r$ so k is unstable.

Question 146. Will there be cases that linearization fail? YES. When $f'(x^*) = 0$.

Example 147. Consider $\dot{x} = -(x-1)^3 = f(x)$. Then the fixed point is $x = 1$. Note that $f'(x) = -3(x-1)^2$, so $f'(1) = 0$. Linearization fails. However, we can use graphical and see that f is decreasing, therefore we move toward the $x = 1$ so $x = 1$ is stable. See figure (55) for visual.

Example 148. Consider $\dot{x} = (x-1)^3 = f(x)$. Then the fixed point is $x = 1$. Note that $f'(x) = 3(x-1)^2$, so $f'(1) = 0$. Linearization fails. However, we can use graphical and see that f is increasing, therefore we move away from the $x = 1$ so $x = 1$ is unstable. See figure (56) for visual.

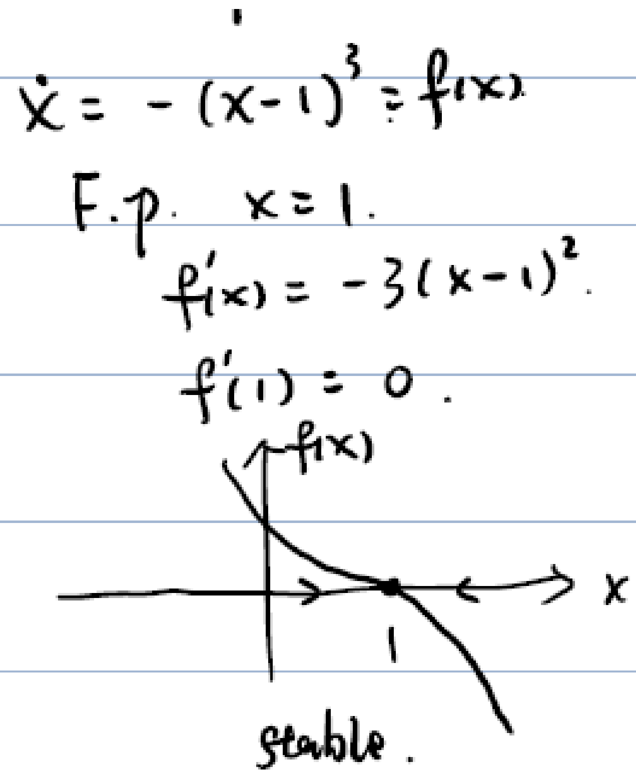

Figure 55: $\dot{x} = -(x-1)^3$

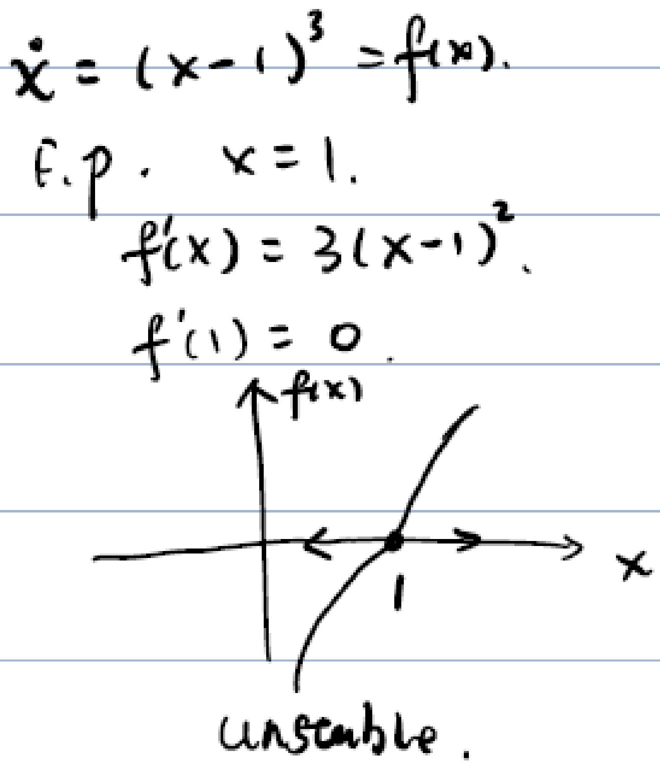

Figure 56: $\dot{x} = (x-1)^3$

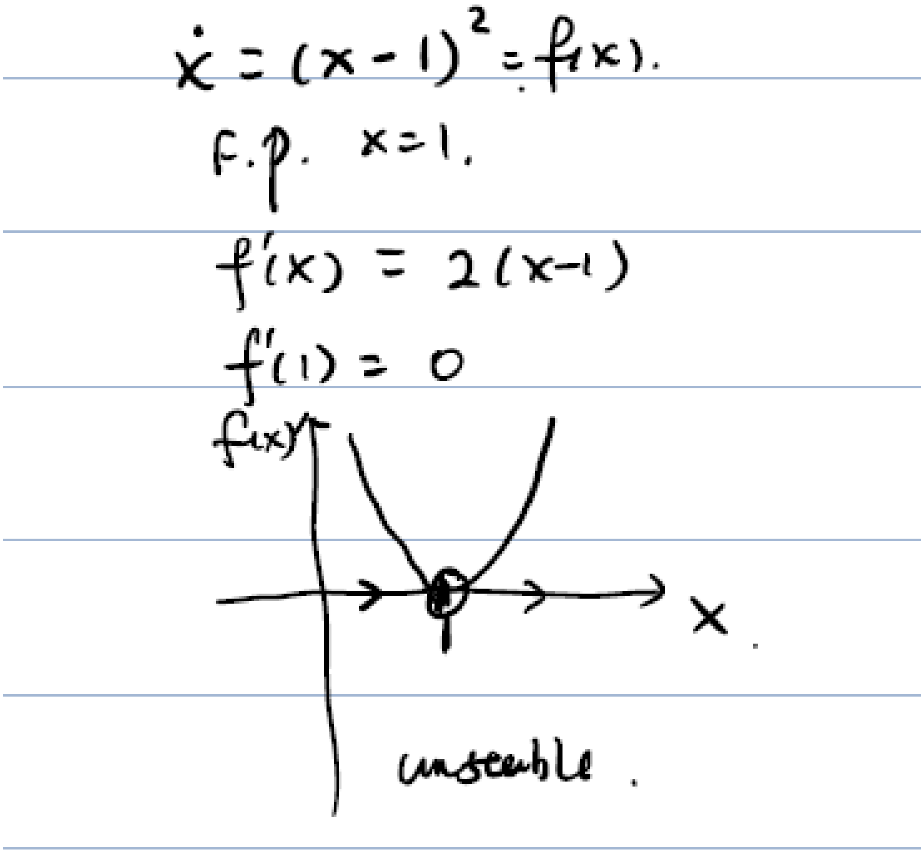

Figure 57: $\dot{x} = (x-1)^3$

Example 149. Consider $\dot{x} = (x-1)^2 = f(x)$. Then the fixed point is $x = 1$. Note that $f'(x) = 2(x-1)$, so $f'(1) = 0$. Linearization fails. However, we can use graphical and see that f is decreasing on the left of $x = 1$ and increasing on the right of $x = 1$, therefore we move toward from the left and move away from the right of $x = 1$, so $x = 1$ is unstable. See figure (57) for visual.

Example 150. The simplest case when $\dot{x} = 0$, we get a line of fixed points.

Theorem 151 (Existence and Uniqueness Theorem)**.** *Consider the initial value problem $\dot{x} = f(x)$ and $x(0) = x_0$. Suppose that $f(x)$ and $f'(x)$ are continuous on an open interval I of the x-axis, adn suppose that x_0 is a point in I. Then the initial value problem has a solution $x(t)$ on some time interval $(-\tau, \tau)$ about $t = 0$ and the solution is unique.*

Example 152. Consider $\dot{x} = 1 + x^2$ with $x(0) = x_0$. Solution exists for all time? Yes, because $f(x) = 1 + x^2$ continuous, with continuous derivative everywhere. So the solution exists and are unique for any x_0. But the theorem does not say the solutions exists for all time. This is because it really depends on the initial condition. Suppose we have the initial condition $x(0) = 0$. Then:

$$\dot{x} = 1 + x^2$$
$$\frac{1}{1+x^2}dx = \int dt$$
$$\arctan(x) = t + c$$

With the initial condition $x(0) = 0$ gives $c = 0$. Therefore $x(t) = \arctan(t)$ is the solution. But this solution only exists for $-\frac{\pi}{2} < t < \frac{\pi}{2}$, as if when $t \to \pm\frac{\pi}{2}$, $x(t) \to \pm\infty$. Outside of the interval, there is no solution to the initial value problem $x_0 = 0$. This is a **blow up phenomenon:** solutions approach infinity in finite time.

3.2 Day 14: Bifurcation of continuous system

3.2.1 Potentials

Definition 153 (potential)**.** Consider $\frac{dx}{dt} = F(x)$, define the potential $V(x)$ by $V'(x) = -F(x)$.

Now if we think of x as a function of t:

$$\frac{d}{dt}V = \frac{dV}{dx}\frac{dx}{dt} = V'(x).(-V'(x)) = -(V'(x))^2 \leq 0$$

Therefore $V(t)$ decreases along trajectories.
At fixed point/equilibrium point, we get $V'(x^*) = 0$ (since it is $\frac{dx}{dt} = 0$). These are called critical points:

1. $V''(x^*) > 0$: minima (stable)
2. $V''(x^*) < 0$: maxima (unstable)
3. $V'(x^*) = 0$: no information

Example 154. Consider $\dot{x} = x - x^3$, so $V'(x) = x^3 - x$. Then $V(x) = -\frac{1}{2}x^2 + \frac{1}{4}x^4$. The stable fixed points are $-1, 1$ and the unstable fixed point are 0. See figure (58).

Remark 155. Note the following:

1. If $\frac{df}{dx}(x^*) \neq 0$, then x^* is called a hyperbolic fixed point.
2. Consider $\dot{x} = f(x, r)$. If x^*, r^* satisfies $f(x^*, r^*) = 0$, a necessary condition for $f(x^*, r^*) = 0$ gives a bifurcation is $\frac{\partial f}{\partial x}(x^*, r^*) = 0$

3.2.2 Bifurcation for continuous system

Definition 156 (bifurcation for continuous system)**.** The qualitative structure of the flow or the phase portrait can change as the parameters are varied. This phenomenon is called bifurcation. The parameter value at which they occur are called bifurcation point.

Example 157 (saddle node bifurcation example)**.** Consider $\dot{x} = r + x^2 = f(x, r)$. Depending on the sign of r, the number of fixed points changed:

1. $r > 0$: there is no fixed point

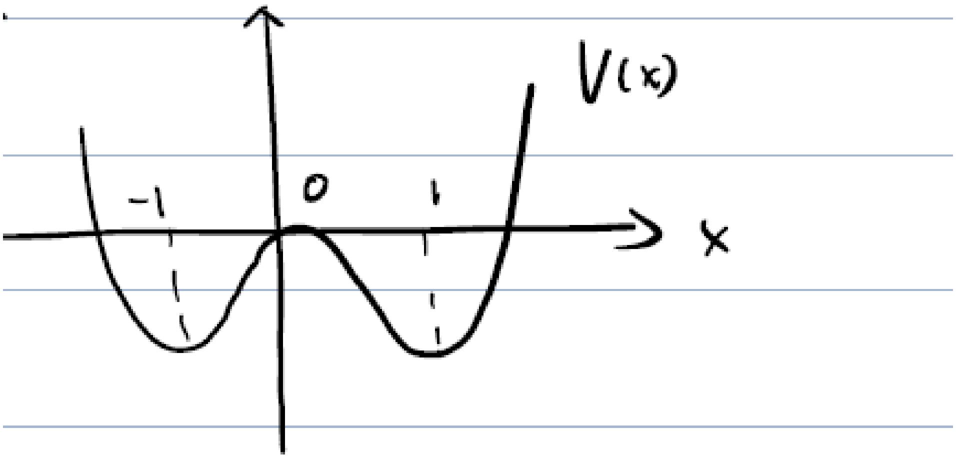

Figure 58: Potential V for $\dot{x} = x - x^3$

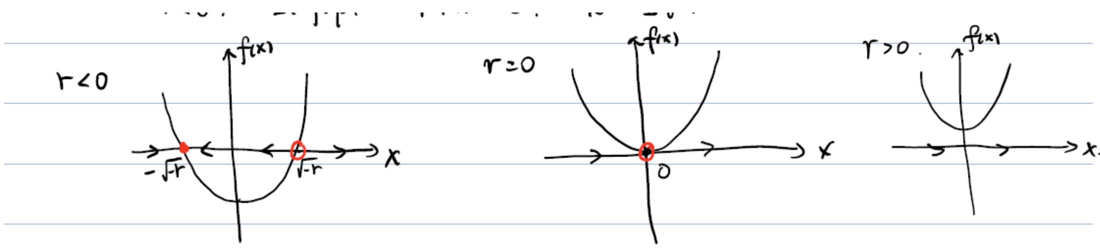

Figure 59: fixed point analysis for $\dot{x} = r + x^2$

2. $r = 0$: there is 1 fixed point at $x = 0$.

3. $r < 0$: there are 2 fixed points: $r + x^2 = 0$, so $x = \pm\sqrt{-r}$.

Note that $f'(x, r) = 2x$ so we can get the stability of these fixed points. See figure (59). Then from these graphs, we see that bifurcation occurs at $r = 0$ in figure (60).

Question 158. Given $\dot{x} = F(x, r)$, which is called **normal form**, under what condition does a saddle node bifurcation occur?

Suppose there exists a point x_e, r_c such that:

$$F(x_e, r_c) = 0, \ \frac{\partial F}{\partial x}(x_c, r_e) = 0$$

with $F_{xx}(x_e, r_c) \neq 0$ and $F_r(x_e, r_c) \neq 0$.
We can linearize near x_e and r_c. Let:

$$x = x_e + y, \ r = r_c + \delta$$

Then:

$$\dot{x} = \dot{y} = F(x_e + y, r_c + y) = F(x_e, r_c) + F_x(x_e, r_c)y + F_r(x_e, r_c)\delta + F_{xx}(x_e, r_c)\frac{y^2}{2} + \dots$$

$$\dot{y} = F_r(x_e, r_c)\delta + F_{xx}(x_e, r_c)\frac{y^2}{2} + \dots \ \text{, since } F(x_e, r_c) = F_x(x_e, r_c) = 0$$

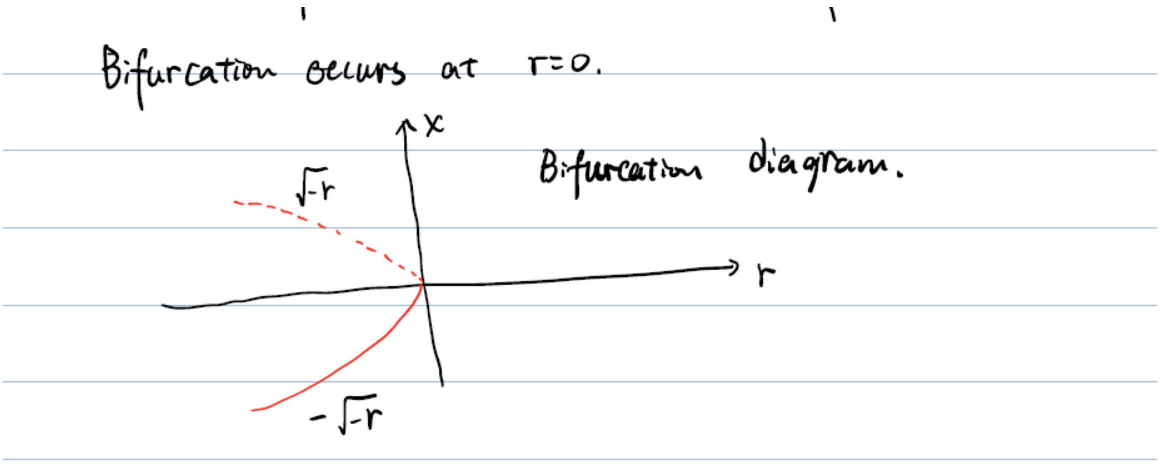

Figure 60: bifurcation for $\dot{x} = r + x^2$ occurs at $r = 0$

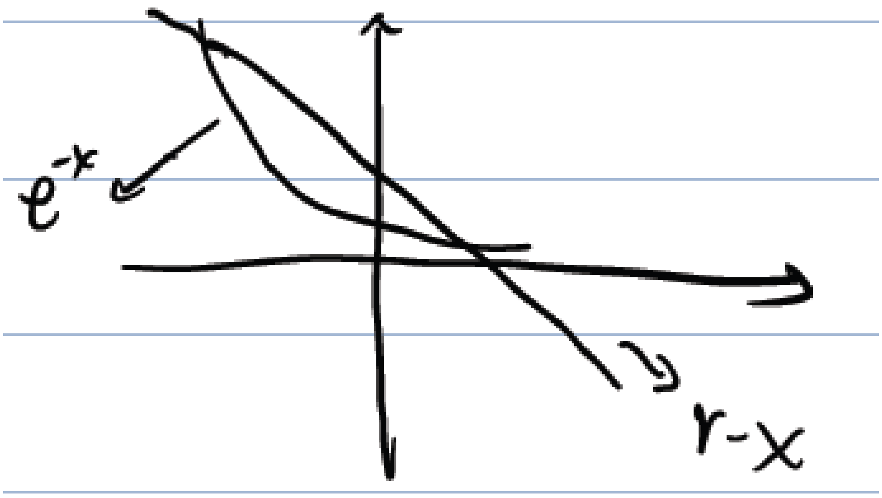

Figure 61: graph of e^{-x} and $r - x$

Define $F_r^0 = F_r(x_e, r_c)$ and $F_{xx}^0 = F_{xx}(x_e, r_c)$. Then we have a local ODE:

$$\dot{y} = F_r^0 \delta + F_{xx}^0 \frac{y^2}{2}$$

Compare this with:

$$\dot{x} = r + x^2$$

Example 159. Consider $\dot{x} = r - x - e^{-x}$. The fixed point are $r - x - e^{-x} = 0$, that is $r - x = e^{-x}$. Graphical analysis with the 2 graphs $y = r - x$ and $y = e^{-x}$ in figure (61) show that when $r \uparrow$, there are 2 roots, and when $r \downarrow$ there is no root.

The bifurcation point is where $F(x, r) = 0$, $F_x(r, x) = 0$:

$$F = 0 \Rightarrow r - x = e^{-x}$$

$$F_x = 0 \Rightarrow -1 + e^{-x} = 0$$

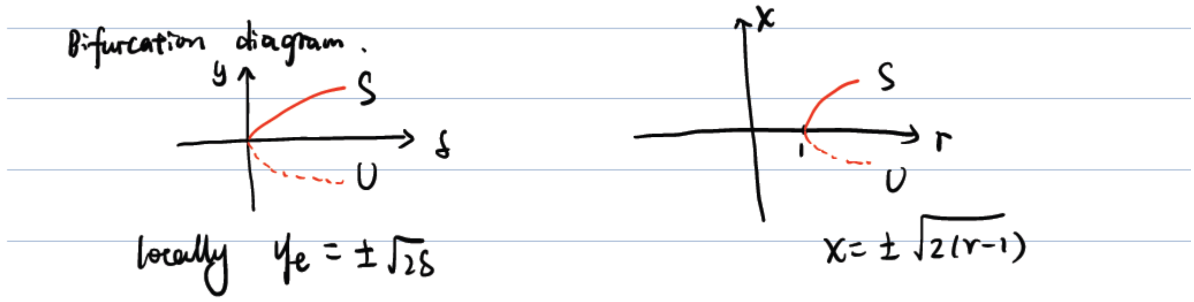

Figure 62: bifurcation for y, δ and x, r

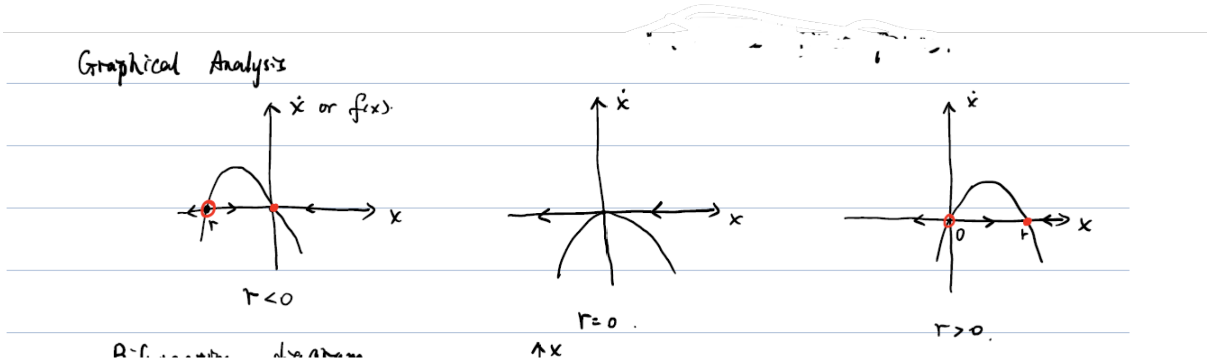

Figure 63: graphical analysis for $\dot{x} = rx - x^2$

Sovling this give $x = 0, r = 1$. In addition, $F_r = 1 \neq 0$ and $F_{xx} = -e^{-x}\Big|_{x=0} = -1$. So near $x = 0$ and $r = 1$ we get $x = y, r = 1 + \delta$. So:

$$\dot{y} = 1.\delta - 1.\frac{y^2}{2} = \delta - \frac{y^2}{2} + \dots$$

We get the two bifurcation diagram for y and for x in figure (62) (the bifurcation for x and r is deduced from bifurcation for y and δ with the relationship $x = y$ but $r = 1 + \delta$)

Example 160 (Transcritical bifurcation)**.** Consider $\dot{x} = rx - x^2 = f(x)$. Fixed point $rx - x^2 = 0$. The fixed points are $x(r - x) = 0$ so $x = 0$ and $x = r$. Note that $f'(x) = r - 2x$. When $r = 0$, we have 1 fixed point. When $r \neq 0$, we have 2 fixed points. See figure (63) for visualization. We also have the bifurcation diagram from that graphical analysis (see figure (64))

Example 161. Consider $\dot{x} = r \ln x + x - 1 = F(x, r)$. Show that the system undergoes a transcritical bifurcation and derive the normal form near the bifurcation.
First note that $x = 1$ is always an equilibrium for any r.

$$F(x, r) = 0 \Rightarrow r \ln x + x - 1 = 0$$

$$F_x(x, r) = 0 \Rightarrow \frac{r}{x} + 1 = 0 \Rightarrow x = -r$$

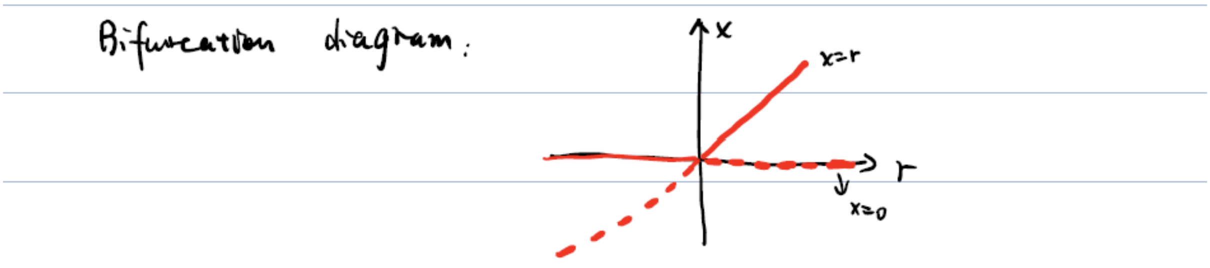

Figure 64: graphical analysis for $\dot{x} = rx - x^2$

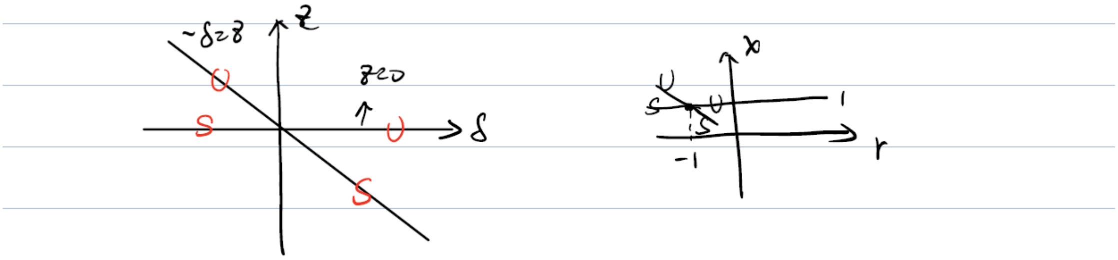

Figure 65: transcritical normal for $\dot{x} = r \ln x + x - 1$ for z, δ and x, r

Then putting it back into $F(x, r) = 0$ and get:

$$x(1 - \ln x) = 1 \Rightarrow 1 - \ln x = \frac{1}{x}$$

Therefore the only solution is $x = 1$ and $r = -1$ (draw out the graphs and see).
Now we look for normal form around $x = 1, r = -1$:
Let $r = -1 + \delta$ and $x = 1 + y$, then:

$$\begin{aligned}
\dot{y} &= (-1 + \delta). \ln(1 + y) + y \\
&= (-1 + \delta)(y - \frac{y^2}{2} + \ldots) + y \\
&= \delta y + \frac{y^2}{2} + O(\delta y^2)
\end{aligned}$$

Let $z = \frac{1}{2} y$:

$$\dot{z} = \delta z + z^2 + O(\delta z^2)$$

Two equilibrium $z = 0$ and $z = -\delta$. See figure (65) for bifurcation diagram in z, δ and in x, r.

Example 162 (Pitchfork bifurcation continuous system)**.** This is common when we have a symmetry. Consider $\dot{x} = rx - x^3 = f(x)$. The fixed point is $rx - x^3 = 0$. So $x(r - x^2) = 0$. Then the fixed points are $x = 0$and $x = \pm\sqrt{r}$. See figure (66) for graphical analysis. From this graphical analysis, we get a supercritical pitchfork bifurcation (see figure (67)).

Example 163. Consider $\dot{x} = rx + x^3$. The fixed point are $x = 0$ and $x = \pm\sqrt{-r}$ and we have 3 fixed points when $r < 0$. See figure () for graphical and supercritical bifurcation diagram.

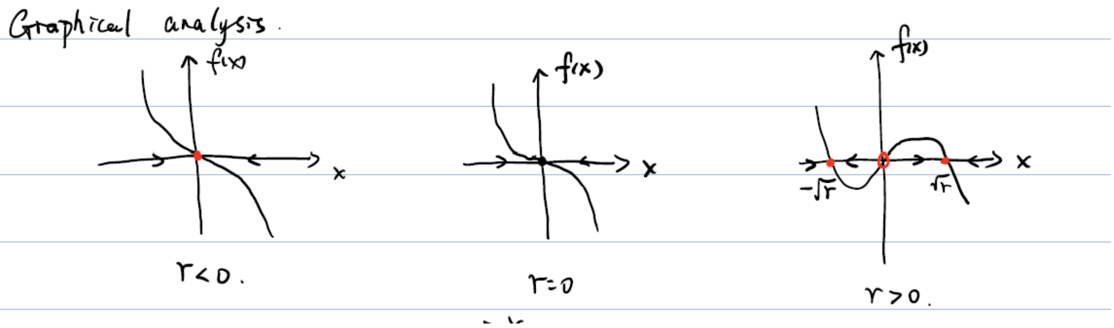

Figure 66: graphical analysis for $\dot{x} = rx - x^3$

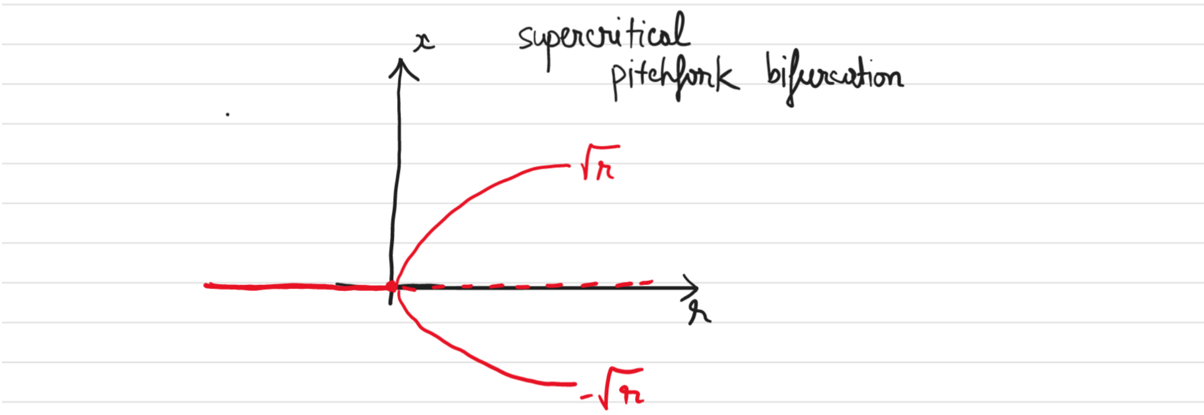

Figure 67: supercritical pitchfork bifurcation for $\dot{x} = rx - x^3$

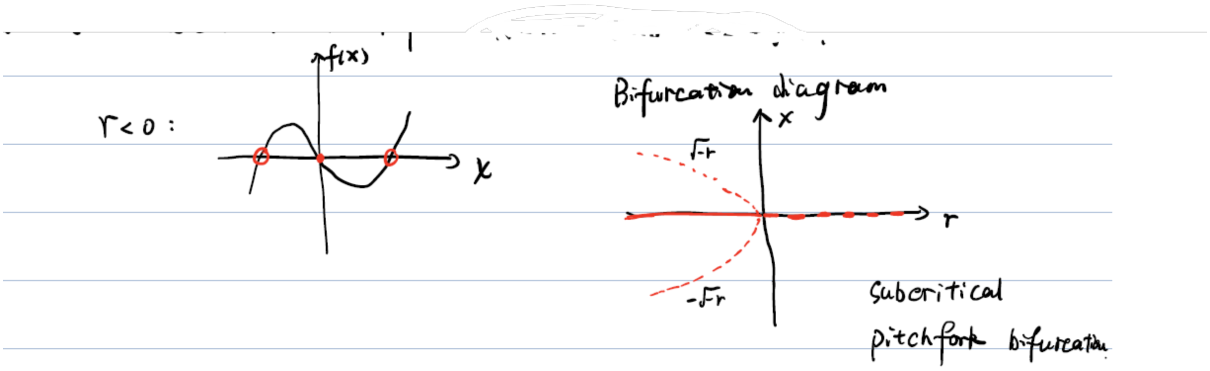

Figure 68: subcritical pitchfork bifurcation for $\dot{x} = rx + x^3$

Theorem 164. *Suppose that $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a smooth function and (x_0, r_0) satisfy the necessary bifurcation condition:*

$$f(x_0, r_0) = 0,\ \frac{\partial f}{\partial x}(x_0, r_0) = 0$$

1. *If $\frac{\partial f}{\partial r}(x_0, r_0) \neq 0$, and $\frac{\partial^2 f}{\partial x^2}(x_0, r_0) \neq 0$, then a saddle-node bifurcation occurs at (x_0, r_0).*
2. *If $\frac{\partial f}{\partial r}(x_0, r_0) = 0$, and $\frac{\partial^2 f}{\partial x \partial r}(x_0, r_0) \neq 0$ and $\frac{\partial^2 f}{\partial x^2}(x_0, r_0) \neq 0$, then a transcritical bifurcation occurs at (x_0, r_0).*
3. *If $\frac{\partial f}{\partial r}(x_0, r_0) = 0$, and $\frac{\partial^2 f}{\partial x \partial r}(x_0, r_0) \neq 0$ and $\frac{\partial^2 f}{\partial x^2}(x_0, r_0) = 0$, and $\frac{\partial^3 f}{\partial x^3}(x_0, r_0) \neq 0$ and $\frac{\partial^2 f}{\partial x^2}(x_0, r_0) = 0$ then a pitchfork bifurcation occurs at (x_0, r_0).*

Example 165. Consider $\dot{x} = r^2 + rx - x^3 = f(x, r)$. Then the fixed points are $f(x, r) = 0$ and $f_x(x, r) = 0$. So:

$$f_x(x, r) = 0 \Rightarrow r - 3x^2 = 0 \Rightarrow r = 3x^2$$

And:

$$\begin{aligned} f(x, r) = 0 \Rightarrow r^2 + rx - x^3 = 0 &\Rightarrow 9x^4 + 3x^2 - x^3 = 0 \\ &\Rightarrow x^3(9x + 2) = 0 \end{aligned}$$

We get 2 pairs $(x = 0, r = 0)$ and $x = -\frac{2}{9}, r = 3.\frac{4}{81} = \frac{4}{27}$

At $x = 0, r = 0$ we calculate:

$$f_r = 2r + x\bigg|_{(0,0)} = 0$$

and

$$f_{xx} = -6x\bigg|_{(0,0)} = 0$$

and

$$f_{xx} = 1 \neq 0,\ f_{xxx} = -6 \neq 0$$

So a pitchfork bifurcation occurs at $r - 0, x = 0$.
At $x = -\frac{2}{9}, r = \frac{4}{27}$:

$$f_r = 2r + x\bigg|_{(-\frac{2}{9}, \frac{4}{27})} = \frac{2.4}{27} - \frac{2}{9} \neq 0$$

and

$$f_{xx} = -6x\bigg|_{x=-\frac{2}{9}} = \frac{12}{9} \neq 0$$

A saddle node bifurcation occurs at $r = \frac{4}{27}$ and $x = -\frac{2}{9}$.

3.3 Day 15 and 16: Two dimensional systems - Chapter 5 Strogatz

Consider

$$\begin{cases} \frac{dx}{dt} &= f(x,y) \\ \frac{dy}{dt} &= g(x,y) \end{cases}$$

equilibrium point (x_0, y_0) satisfies:

$$\begin{cases} f(x_0,y_0) &= 0 \\ g(x_0,y_0) &= 0 \end{cases}$$

Let $x = x_0 + \tilde{x}$, and $y = y_0 + \tilde{y}$ with $\tilde{x}, \tilde{y} \ll 1$. Substitute them into the original system and expand the RHS using Taylor expansion:

$$\begin{aligned} &= f_x^0\tilde{x} + f_y^0\tilde{y} + \text{ higher order terms} \\ \frac{d\tilde{y}}{dt} &= g_x^0\tilde{x} + g_y^0\tilde{y} + \text{ higher order terms} \end{aligned}$$

Linearization system:

$$\begin{pmatrix} \frac{d\tilde{x}}{dt} \\ \frac{d\tilde{y}}{dt} \end{pmatrix} = \begin{pmatrix} f_x^0 & f_y^0 \\ g_x^0 & g_y^0 \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = J \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix}$$

where

$$\overrightarrow{x} = \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix}$$

Solving the ODE as before we got:

$$\overrightarrow{x} = e^{\lambda t}\overrightarrow{v}$$

where $(J - \lambda I)\overrightarrow{v} = 0$

Question 166. We have these questions:

1. Classify the equilibria
2. Does the classification persists under small non-linear terms?
3. How does the system look like near equilibria?

3.3.1 Analysis of two dimensional system based on Jacobian matrix

Now consider the Jacobian matrix

$$J = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

eigenvalues λ satisfy:

$$\begin{aligned} (a-\lambda)(d-\lambda) - bc = 0 \\ \lambda^2 - (a+d)\lambda + ad - bc = 0 \end{aligned}$$

$$\lambda_\pm = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}$$

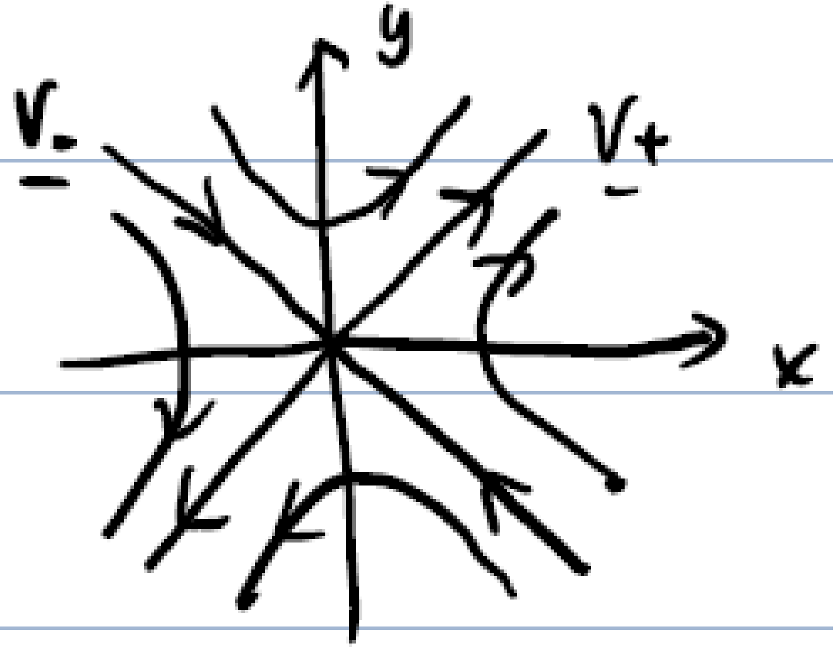

Figure 69: saddle point two dimensional system

where $\Delta = \pi J = a + d$ and $\Delta = \det J = ad - bc$. Then we will have the $\overrightarrow{x}$ are:

$$\overrightarrow{x} = c_1 e^{\lambda_+ t} \overrightarrow{v_+} + c_2 e^{\lambda_- t} \overrightarrow{v_-}$$

Now depends on the values $\lambda_\pm$ we have these cases:

1. If both $\lambda_\pm$ are real. In addition, if $\lambda_+ > 0$ and $\lambda_- < 0$. We need $\Delta < 0$. Then:

   $$\overrightarrow{x} = c_1 e^{\lambda_+ t} \overrightarrow{v_+} + c_2 e^{\lambda_- t} \overrightarrow{v_-} \xrightarrow{t \to \infty} c_1 e^{\lambda_+ t} \overrightarrow{v_+}$$

   This we will have a **saddle point**. See figure (69) for visual.

2. If $\lambda_\pm$ are real but $\lambda_- < \lambda_+ < 0$, then we need $\tau^2 - 4\Delta > 0$. Note that this mean $\tau < 0$ and $\Delta > 0$. In addition, we note that $|\lambda_+| < |\lambda_-|$, and that $\overrightarrow{v_+}$ is the slow eigendirection, while $\overrightarrow{v_-}$ is the fast direction. We also have a **stable** node case, see figure (70).

3. If $\lambda_\pm$ are real but $0 < \lambda_- < \lambda_+$, then we need $\tau^2 - 4\Delta > 0$. Note that this mean $\tau > 0$ and $\Delta > 0$. In this case we have **unstable** node. See figure (71):

4. If $\lambda_\pm$ are complex with $\Re(\lambda_\pm) > 0$. Let's say $\lambda_\pm = \alpha \pm i\omega$ with $\alpha > 0$. Then recall:

   $$\overrightarrow{x} = c_1 e^{\lambda_+ t} \overrightarrow{v_+} + c_2 e^{\lambda_- t} \overrightarrow{v_-}$$

   Then by Euler's formula $e^{i\omega t} = \cos(\omega t) + i \sin(\omega t)$:

   $$e^{\lambda_\pm t} = e^{(\alpha \pm i\omega)t} = e^{\alpha t} e^{\pm i\omega t}$$

   Therefore $\overrightarrow{x}$ involves $e^{\alpha} \cos(\omega t)$ and $e^{\alpha t} \sin(\omega t)$ **two unstable spiral nodes** see figure (72):

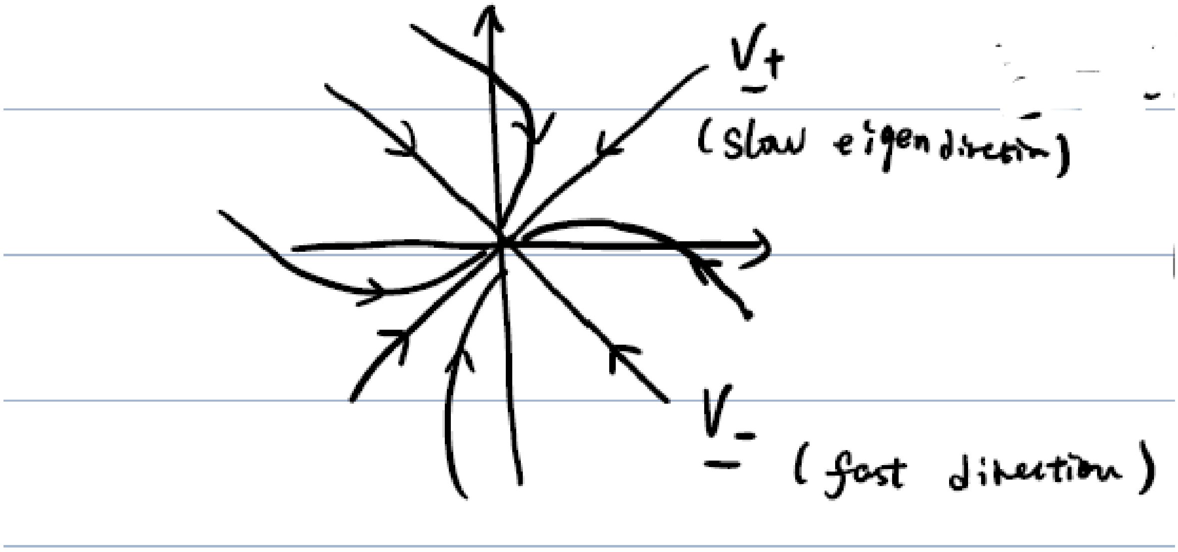

Figure 70: stable node point two dimensional system

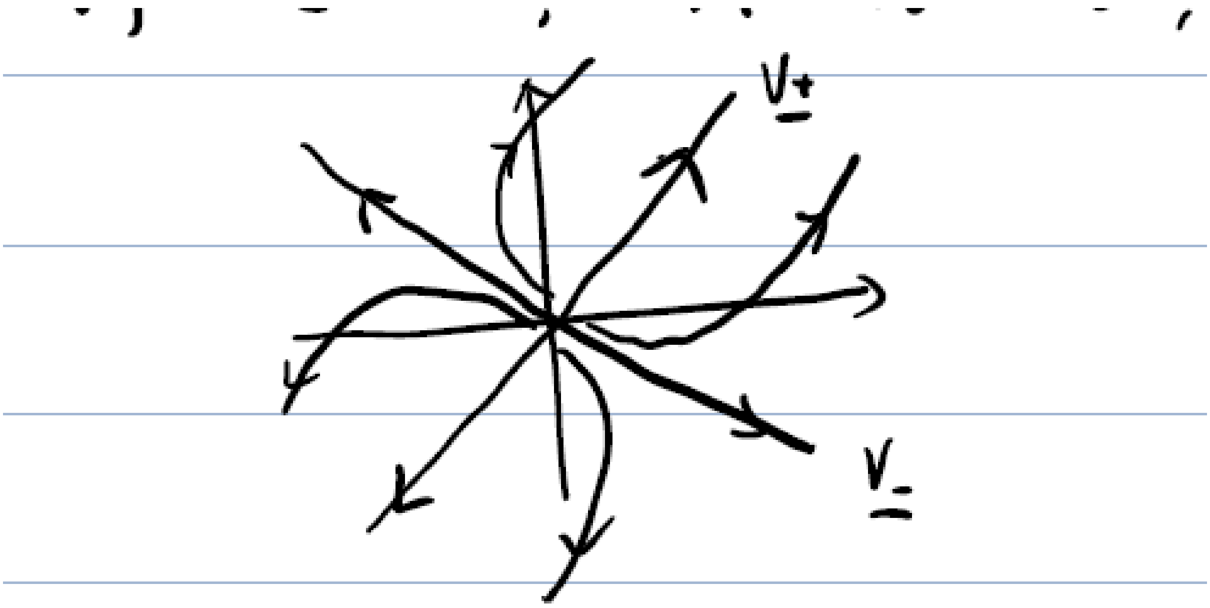

Figure 71: unstable node point two dimensional system

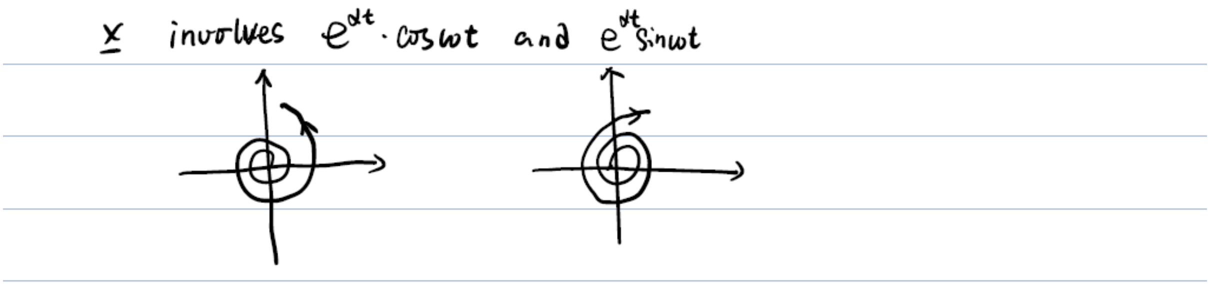

Figure 72: unstable spiral for two dimensional system

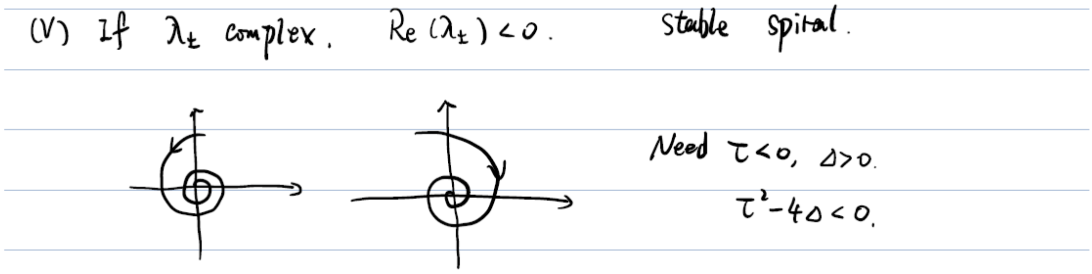

Figure 73: stable spiral for two dimensional system

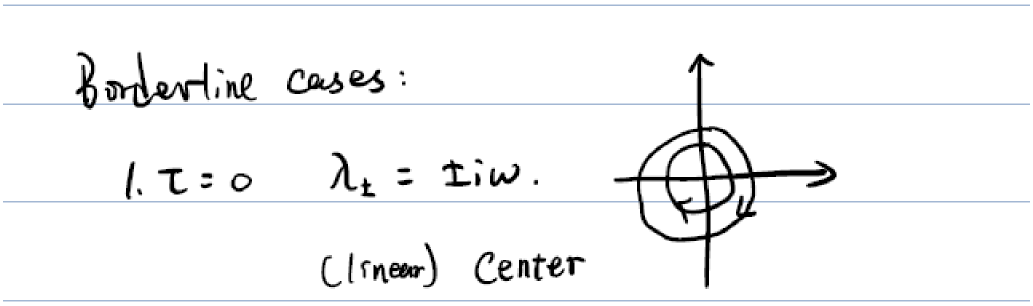

Figure 74: linear center two dimensional system

5. If $\lambda_\pm$ complex with $\Re(\lambda_\pm) < 0$. In this case we will get **stable spiral**. We need $\tau < 0$ and $\Delta > 0$ so $\tau^2 - 4\Delta < 0$. See figure (73):
6. The first borderline case: $\tau = 0$: then $\lambda_\pm = \pm i\omega$ (purely imaginary) We will get a (linear) central as in figure (74)
7. The second borderline case: $\Delta = 0$: At least one of the eigenvalues is 0. In the linear system, $(0,0)$ is not an isolated fixed point (it is a line or a plane of fixed point).
8. $\tau^2 = 4\Delta$: we get equal eigenvalue. There are two possible cases:
   (a) 2 independent eigenvectors: they will span the plan. So any vector is an eigenvector $\overrightarrow{x(t)} = e^{\lambda t}\overrightarrow{x_0}$. All trajectory are straight lines through the origin. So the fixed point is a **star node**. See figure (75)
   (b) 1 eigenvector (eigenspace is 1-dimensional): The fixed point is a degenerate node, see figure (76)

See figure (77) for a summary of all the cases.

Remark 167. The classifications of an equilibrium point for nonlinear system is exactly the same for the linearized problem except for borderline cases. For instance, the center may be a stable/unstable spiral or remain a center.

Example 168. Consider:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

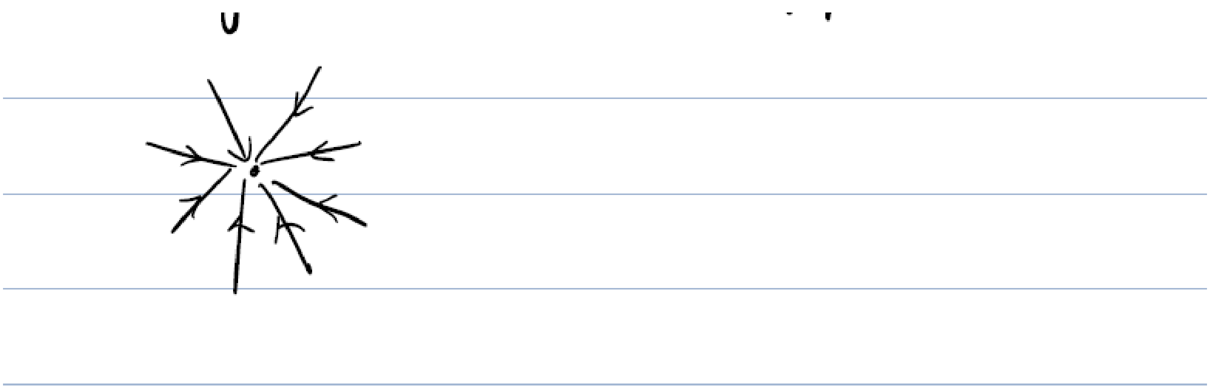

Figure 75: star node two dimensional system

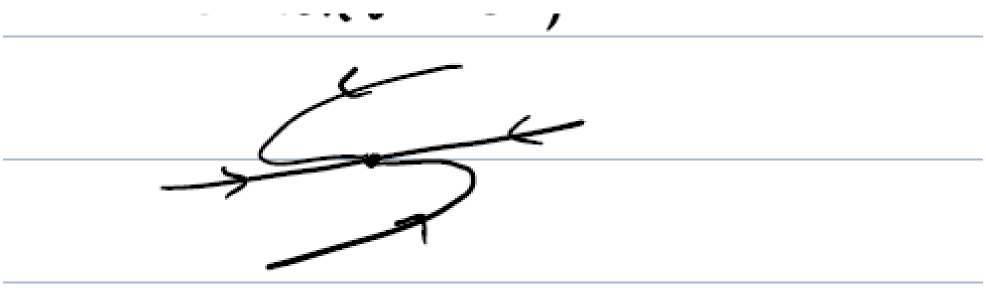

Figure 76: degenerate node two dimensional system

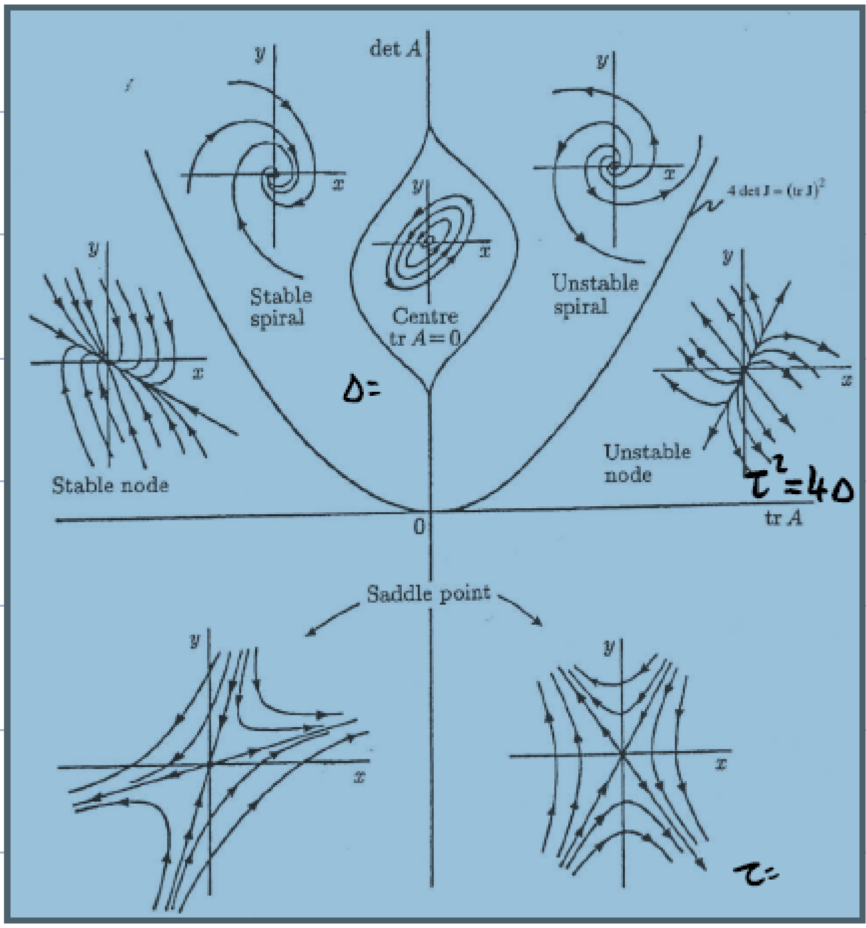

Figure 77: two dimensional system summary

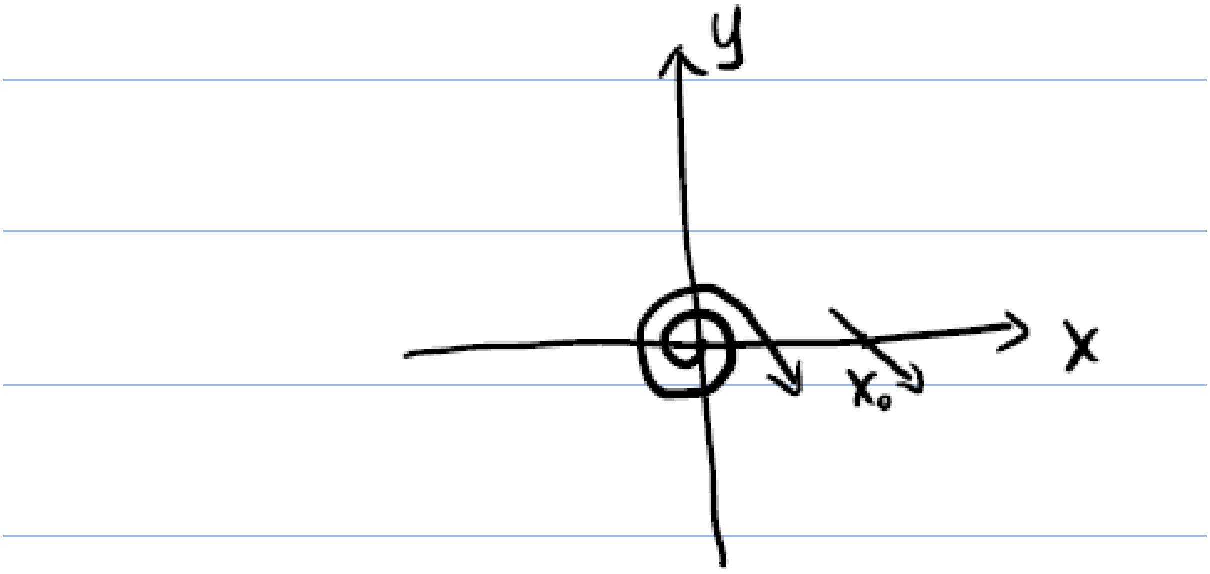

Figure 78: two dimensional system example 1

With the matrix

$$J = \begin{pmatrix} 2 & 3 \\ -3 & 2 \end{pmatrix}$$

We get $\tau = \mathrm{tr}J = 4 > 0$ and $\Delta = \det J = 13 > 0$, and $\tau^2 - 4\Delta = 16 - 4.13 < 0$ so we have **unstable spiral**. The fixed point is at $(0,0)$. Also, solve the eigenvalue and get $\lambda = 2 \pm 3i$. Let

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_0 \\ 0 \end{pmatrix} \quad x_0 > 0$$

Then:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 2x_0 \\ -3x_0 \end{pmatrix}$$

Since $2x_0 > 0$ and $-3x_0 < 0$ we get a clockwise rotation as in figure (78)

Example 169. Consider:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 5 & -4 \\ -8 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

With the matrix

$$J = \begin{pmatrix} 5 & -4 \\ -8 & 1 \end{pmatrix}$$

We get $\tau = \mathrm{tr}J = 6 > 0$ and $\Delta = \det J = 5 - 32 < 0$ so this is a saddle point. The eigenvalues are:

$$\begin{aligned} \lambda^2 - \tau\lambda + \Delta &= 0 \\ \lambda^2 - 6\lambda - 27 &= 0 \\ \lambda &= -3, 9 \end{aligned}$$

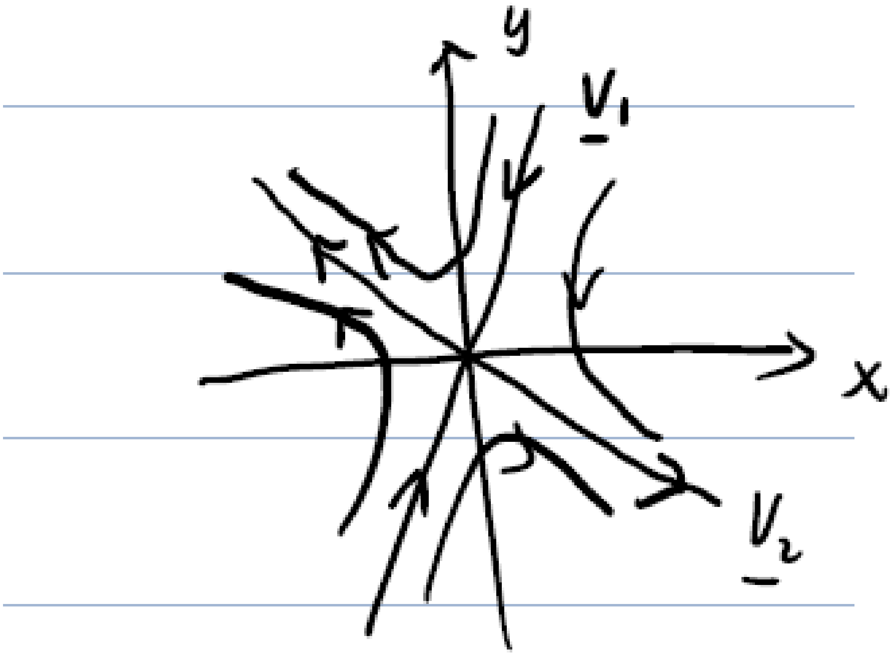

Figure 79: two dimensional system example 2

1. With $\lambda_1 = -3$: We solve:
$$(J + 3I)\overrightarrow{v} = \overrightarrow{0}$$
And get:
$$\begin{pmatrix} 8 & -4 \\ -8 & 4 \end{pmatrix} \overrightarrow{v} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
This gives
$$\overrightarrow{v_1} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

2. Wtth $\lambda_2 = 9$ we solve:
$$(J - 9I)\overrightarrow{v} = \overrightarrow{0}$$
And get:
$$\begin{pmatrix} -4 & -4 \\ -8 & -8 \end{pmatrix} \overrightarrow{v} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
This gives
$$\overrightarrow{v_1} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

See figure (79) for the diagram.

Example 170. Consider $\dot{x} = \alpha y$ and $\dot{y} = -\alpha x$, for $\alpha > 0$. The fixed point is at $(0, 0)$ and the Jacobian matrix is
$$J = \begin{bmatrix} 0 & \alpha \\ -\alpha & 0 \end{bmatrix}$$

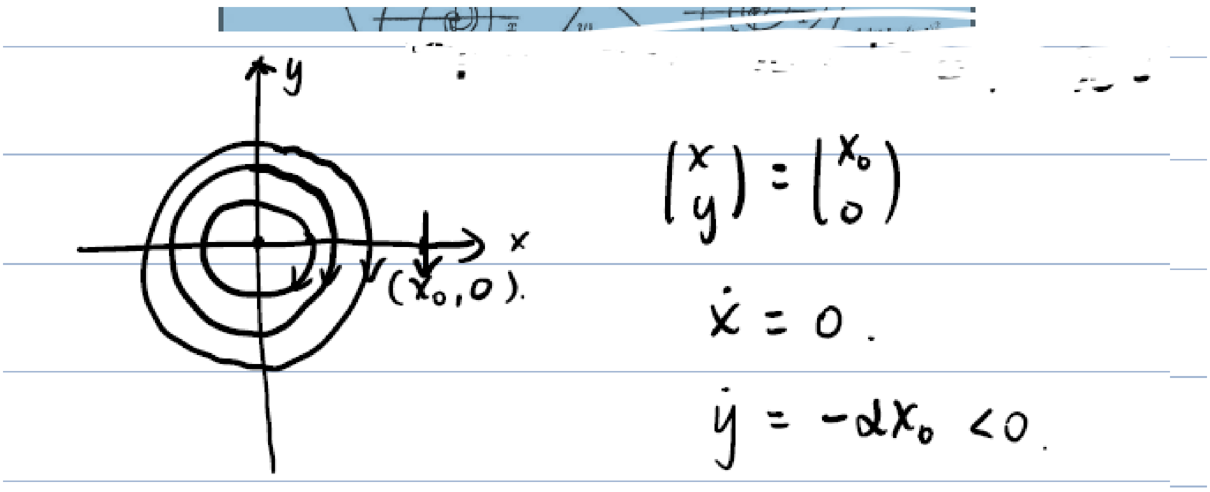

Figure 80: two dimensional system example 3

The eigenvalues $\lambda^2 + \alpha^2 = 0$ so $\lambda = \pm i\alpha$. The $\tau = \operatorname{tr} J = 0$ and $\Delta = \det J = \alpha^2 > 0$. Therefore we got a circular case rotation down, see figure (80). This is because suppose

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_0 \\ 0 \end{pmatrix}$$

Then $\dot{x} = 0$ and $\dot{y} = -\alpha x_0 < 0$ pointing down.

3.4 Day 17: More examples of two dimensional continuous system

3.4.1 More important examples: conservative and Hamiltonian

Example 171 (Conservative System)**.** A particle of mass m moving along the $x-$ axis, subject to a nonlinear force $F(x)$. The equation of motion is

$$m\ddot{x} = F(x)$$

F is independent of $\dot{x}$ and t, no damping or friction, no time-dependent driving force. Let $V(x)$ be the potential energy, $F(x) = -\frac{dV}{dx}$. Denote $V'(x) = \frac{dV}{dx}$ we have:

$$m\ddot{x} + V'(x) = 0$$

To find the equilibrium and its stability, let:

$$\dot{x} = y \text{ and } \dot{y} = -V'(x)$$

Also, let $x = x_0$ and $y = 0$ be an equilibrium. Then linearization get:

$$\begin{cases} x &= x_0 + \tilde{x} \\ y &= 0 + \tilde{y} \end{cases}$$

So

$$\begin{pmatrix} \dot{\tilde{x}} \\ \dot{\tilde{y}} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -V''(x_0) & 0 \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix}$$

The eigenvalue then must satisify $\lambda^2 + V''(x_0) = 0$.

1. If $V''(x_0) < 0$, this is a maximum. So $\lambda_{pm} = \pm\sqrt{-V''(x_0)}$: saddle point.
2. If $V''(x_0) > 0$, this is a minimum. So $\lambda_{pm} = \pm i\sqrt{V''(x_0)}$: center.

Linearization theory predicts a center. But is there really a closed orbit for the nonlinear problem?
Multiply both sides of

$$\dot{x} = y \text{ and } \dot{y} = -V'(x)$$

by $\dot{x}$ and $\dot{y}$ we get:

$$\begin{aligned} \dot{x}\dot{y} &= y\dot{y} \\ \dot{y}\dot{x} &= -V'(x)\dot{x} \end{aligned}$$

Therefore:

$$\begin{aligned} y\dot{y} + V'(x)\dot{x} &= 0 \\ \Rightarrow \frac{d}{dt}(\frac{1}{2}y^2 + V(x)) &= 0 \end{aligned}$$

Then the total energy $E = \frac{1}{2}\dot{x}^2 + V(x)$ is a constant of time. We noted E the conserved quantity. Motion is on the level curves of E. As $V(x)$ has a local min at $x = x_0$, E is locally convex, closed orbits persist in the nonlinear system.

Theorem 172. *Consider the system $\dot{x} = F(x)$, where $\overrightarrow{x} = (x, y) \in \mathbb{R}^2$ and $\overrightarrow{F(\overrightarrow{x})}$ is continuously differentiable. Suppose there exists a conserved quantity $E(\overrightarrow{x})$ and suppose $\overrightarrow{x_0}$ is an isolated fixed points. If $\overrightarrow{x_0}$ is a local minimum of E, then all trajectories sufficiently close to $\overrightarrow{x_0}$ are closed.*

Example 173 (Nonlinear terms might change a linear center)**.** Consider the system with β constant:

$$\begin{cases} \dot{x} &= y + \beta x(x^2 + y^2) \\ \dot{y} &= -x + \beta y(x^2 + y^2) \end{cases}$$

Find and classify the fixed point. The $(0, 0)$ is a fixed point. Note the Jacobian matrix:

$$J = \begin{pmatrix} 3\beta x^2 + \beta y^2 & 1 + 2\beta xy \\ -1 + 2\beta xy & \beta x^2 + 3\beta y^2 \end{pmatrix}\Bigg|_{(0,0)} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Then $\mathrm{tr} J = 0$ and $\det J = 1 > 0$, so eigenvalues $\lambda = \pm i$. Thus $(0, 0)$ is a (linear) center. The linearized theory predicts a center at the point $(0, 0)$. Do the nonlinear terms pertube the center to become spiral?
To analyze the nonlinear system, we use polar coordinates:

$$\begin{cases} x &= r\cos\theta \\ y &= r\sin\theta \end{cases}$$

where $r^2 = x^2 + y^2$ and $\theta = \arctan(\frac{y}{x})$. Then:

$$\begin{aligned} 2r\dot{r} &= 2x\dot{x} + 2y\dot{y} \\ \Rightarrow r\dot{r} &= x(y + \beta x(x^2 + y^2)) + y(-x + \beta y(x^2 + y^2)) \\ r\dot{r} &= \beta(x^2 + y^2)(x^2 + y^2) = \beta(x^2 + y^2)^2 = \beta r^4 \\ \Rightarrow \dot{r} &= \beta r^3 \end{aligned}$$

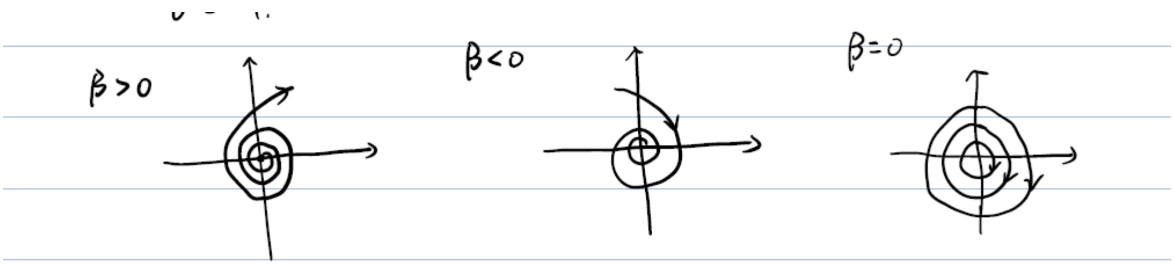

Figure 81: nonlinear terms might change a linear center

From the definition $\theta = \arctan(\frac{y}{x})$ we get:

$$\dot{\theta} = \frac{\frac{\dot{y}}{x} - \frac{y\dot{x}}{x^2}}{1 + \frac{y^2}{x^2}} = \frac{x\dot{y} - \dot{x}y}{x^2 + y^2} = -1$$

So we get the system:

$$\begin{cases} \dot{r} & = \beta r^3 \\ \dot{\theta} & = -1 \end{cases}$$

And we got three cases for the graphical (depends on value β), see figure (81).

Definition 174 (Hamiltonian system)**.** A system is Hamiltonian iff there is some $H = H(x, y)$ such that

$$\frac{d}{dt} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{\partial H}{\partial y} \\ -\frac{\partial H}{\partial x} \end{pmatrix}$$

The function H is called **the Hamiltonian function** for the system.

Example 175. Show that the system

$$\begin{cases} \frac{dx}{dt} & = -2x - 3y^2 \\ \frac{dy}{dt} & = -3x^2 + 2y \end{cases}$$

is Hamiltonian.
Need to find $H(x, y)$:
Assume

$$\frac{\partial H}{\partial y} = -2x - 3y^2$$
$$\Rightarrow H(x, y) = -2xy - y^3 + f(x)$$

Next recall that:

$$\frac{\partial H}{\partial x} = -(-3x^2 + 2y) = 3x^2 - 2y$$

Calculate

$$\frac{\partial H}{\partial x} = -2y + f'(x)$$

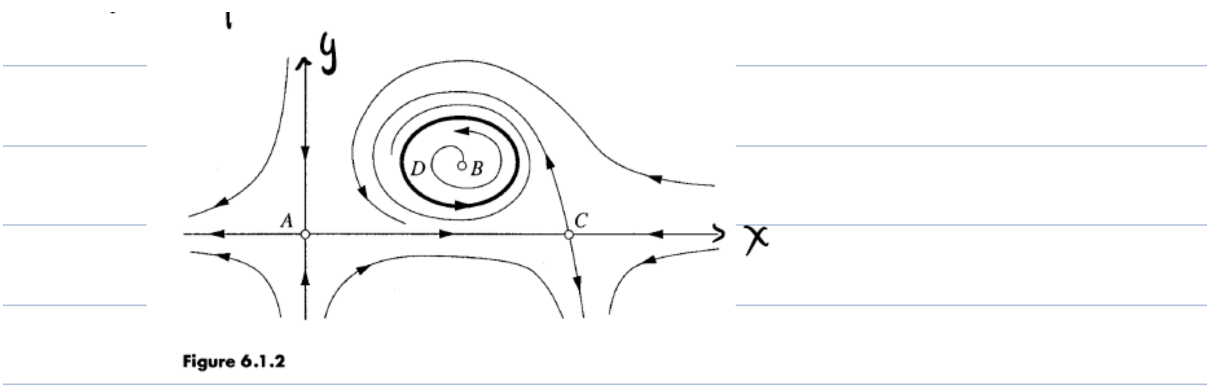

Figure 82: Global phase portrait idea

We get $f'(x) = 3x^2$, so $f(x) = x^3$. Thus:

$$H(x, y) = x^3 - y^3 - 2xy$$

For the Hamiltonian systems

$$\frac{dH}{dt} = H_x\dot{x} + H_y\dot{y} = 0$$

Therefore $H(x, y)$ is constant along any solution curve.
Exercise: Verify that if $H(x, y)$ has a minima/maxima at $\overrightarrow{x_0}$ where $\overrightarrow{x_0}$ is a center.

3.4.2 Global phase portrait

Consider the system:

$$\begin{cases} \dot{x} &= f(x, y) \\ \dot{y} &= g(x, y) \end{cases}$$

We want to analyze the qualitative behavior of the solution in the phase plane. See figure (82)

The features that we want to obtain are:

1. Fixed points A,B, C
2. Closed orbits, periodic solution: $\overrightarrow{x}(t) = \overrightarrow{(t + \tau)}$ where τ is a constant.
3. Arrangement of trajectories near the fixed points and closed orbits.
4. Stability of fixed points and closed orbits.

Example 176. Consider the system:

$$\begin{cases} \dot{x} &= x + e^{-y} \\ \dot{y} &= -y \end{cases}$$

Show the global phase portrait.
The fixed points is $x + e^{-y} = 0$ and $-y = 0$. Therefore we get $(-1, 0)$ is the only

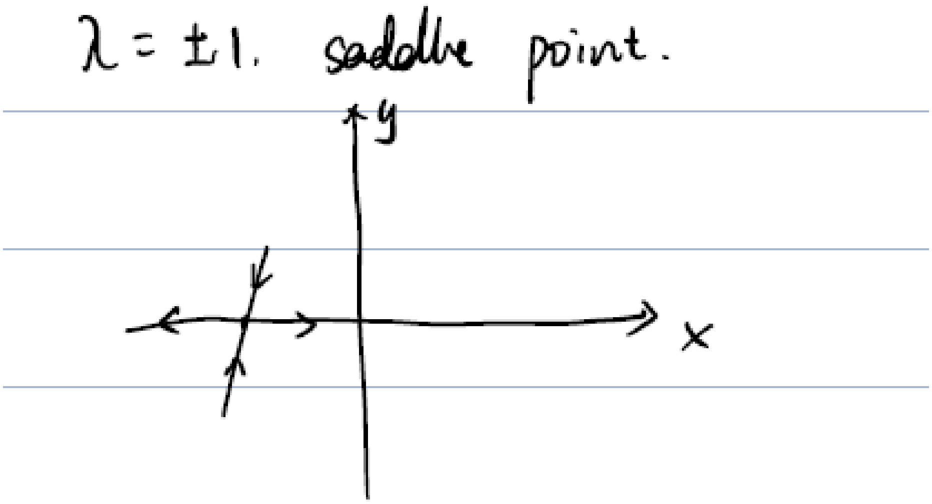

Figure 83: Global phase portrait fixed point

equilibrium point.
For stability, the Jacobian matrix is:

$$J = \begin{pmatrix} 1 & -e^{-y} \\ 0 & -1 \end{pmatrix}\Bigg|_{(-1,0)} = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}$$

The eigenvalue is $\lambda = \pm 1$, so we get saddle point.

1. When $\lambda = 1$ we get the eigenvector

$$\overrightarrow{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

2. When $\lambda = -1$ we get the eigenvector

$$\overrightarrow{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

See figure (83) for the **phase portrait (with eigenvectors near the fixed point)**

Definition 177 (Nullcline)**.** The **Nullcline** is the curve when either $\dot{x} = 0$ and $\dot{y} = 0$.

So the nullclines for this example is:

$$\begin{aligned} \dot{x} &= 0 \\ x + e^{-y} &= 0 \\ x &= e^{-e^{-y}} \end{aligned}$$

Now for the other $\dot{y} = -y$:

$$\begin{aligned} -y &= 0 \\ y &= 0 \end{aligned}$$

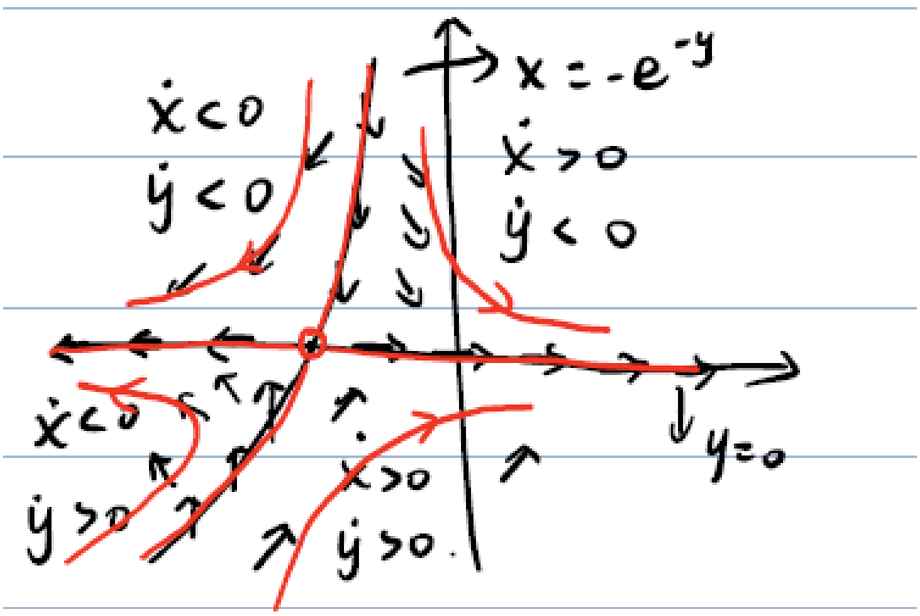

Figure 84: Nullclines

So

$$\dot{x} = x + e^{-y} = x + 1 = \begin{cases} < 0 & x < -1 \\ > 0 & x > -1 \end{cases}$$

Nullcline indicate where the trajectories are purely horizontal or vertical. It partitions the plane into regions where $\dot{x}$ and $\dot{y}$ have different signs. Look at figure (84) for the nullclines and phase portrait.

Remark 178. Assume the function $\overrightarrow{F}(x, y)$ is sufficiently smooth, the solution trajectories cannot cross (uniqueness). Trajectories can meet at a fixed point as $t \to \infty$, if stable. If we consider any trajectory starting inside a closed curve C, C is a trajectory, it remains in C for all time.

3.5 Day 18: Population Problems

Definition 179 (Population problems)**.** Let x, y denote the population of two species ≥ 0. There are these types of model:

1. **Predator-prey with overcrowding**: x is predator and y is prey, so

$$\begin{cases} \dot{x} &= (-\alpha - ax + by)x \\ \dot{y} &= (\beta - cx - dy)y \end{cases}$$

2. **The competition model**: with $\alpha, \beta, a, b, c, d > 0$:

$$\begin{cases} \dot{x} &= (\alpha - ax - by)x \\ \dot{y} &= (\beta - cx - dy)y \end{cases}$$

3. **The cooperation model**

$$\begin{cases} \dot{x} & = (\alpha - ax + by)x \\ \dot{y} & = (\beta + cx - dy)y \end{cases}$$

Example 180. Consider the system:

$$\begin{cases} \dot{x} & = x(3 - x - 2y) = f(x, y) \\ \dot{y} & = y(2 - x - y) = g(x, y) \end{cases}$$

The equilibrium points is when:

$$\begin{cases} f(x, y) & = 0 \\ g(x, y) & = 0 \end{cases}$$

Therefore:

$$\begin{cases} x(3 - x - 2y) & = 0 \\ y(2 - x - y) & = 0 \end{cases}$$

Then $x = 0$ or $3 - x - 2y = 0$, or $y = 0$ or $2 - x - y = 0$. We get the following four equilibrium points: $(0, 0), (0, 2), (3, 0), (1, 1)$.
To classify these fixed points, we need the Jacobian matrix:

$$J = \begin{pmatrix} 3 - 2x - 2y & -2x \\ -y & 2 - x - 2y \end{pmatrix}$$

We identify each cases of fixed points:

1. For $(0, 0)$, the Jacobian matrix is

$$J = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$$

   and it has 2 eigenvalues $\lambda = 3, 2$: unstable node. The eigenvectors are:

   (a) $\lambda = 3$, the eigenvector is

$$\overrightarrow{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

   (b) $\lambda = 2$, the eigenvector is

$$\overrightarrow{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

   The portrait for this fixed point $(0, 0)$ is in figure (85)

2. for fixed point $(0, 2)$, the Jacobian is

$$J = \begin{pmatrix} -1 & 0 \\ -2 & -2 \end{pmatrix}$$

   and it has 2 eigenvalues $\lambda = -1, -2$: stable node. The eigenvectors are:

   (a) $\lambda = -1$, the eigenvector is

$$\overrightarrow{v} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

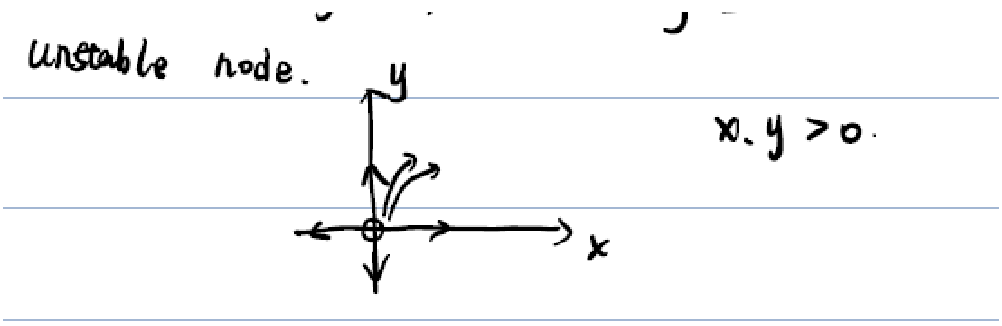

Figure 85: unstable node population model

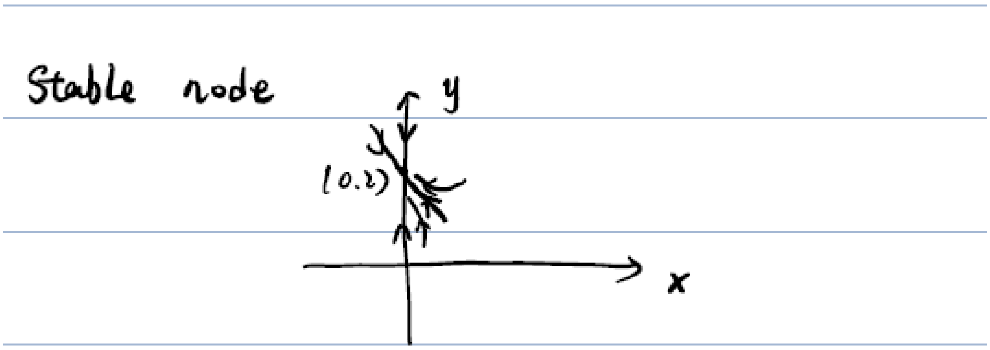

Figure 86: stable node population model

(b) $\lambda = -2$, the eigenvector is

$$\vec{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The portrait for this fixed point $(0, 2)$ is in figure (86)

3. for fixed point $(3, 0)$, the Jacobian is

$$J = \begin{pmatrix} -3 & -6 \\ 0 & -1 \end{pmatrix}$$

and it has 2 eigenvalues $\lambda = -3, -1$: stable node. The eigenvectors are:

(a) $\lambda = -1$, the eigenvector is

$$\vec{v} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

(b) $\lambda = -3$, the eigenvector is

$$\vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

The portrait for this fixed point $(3, 0)$ is in figure (87)

4. for fixed point $(1, 1)$, the Jacobian matrix is:

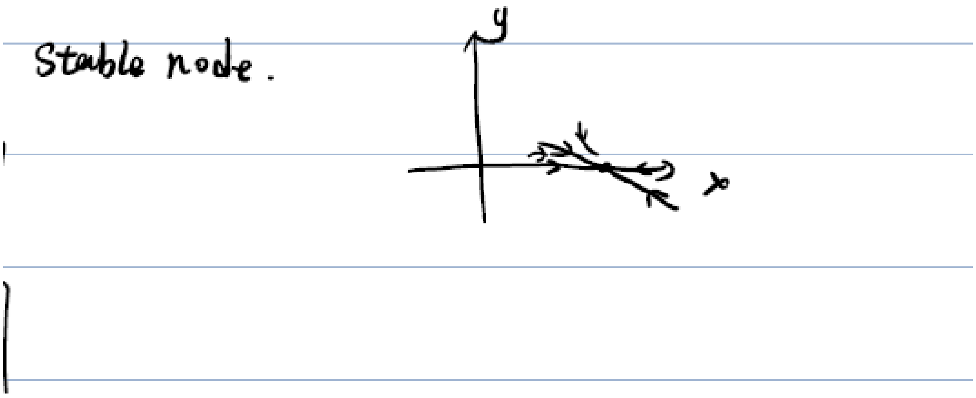

Figure 87: stable node population model

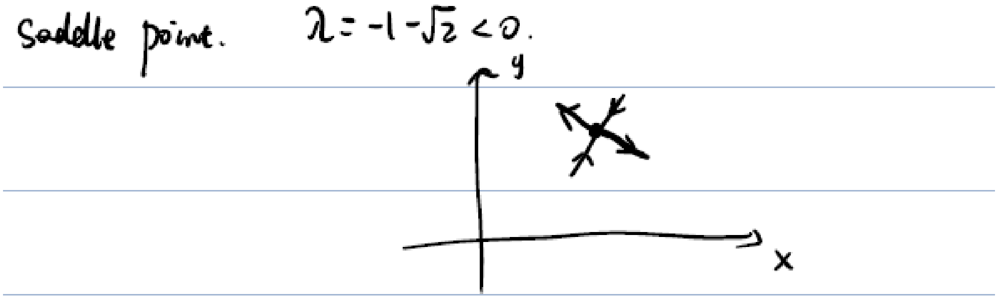

Figure 88: saddle node population model

$$J = \begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix}$$

The eigenvalues are $(\lambda+1)^2 - 2 = 0$, and this gives two eigenvalues $\lambda = -1 \pm \sqrt{2}$.There is saddle point at $\lambda = -1+\sqrt{2}$.

The eigenvectors are:

(a) $\lambda = -1+\sqrt{2}$, the eigenvector is

$$\overrightarrow{v} = \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix}$$

(b) $\lambda = -1-\sqrt{2}$, the eigenvector is

$$\overrightarrow{v} = \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix}$$

The portrait for this fixed point $(3,0)$ is in figure (88)

Now we get the summarizing results for fixed points in figure (89)

Recall also the

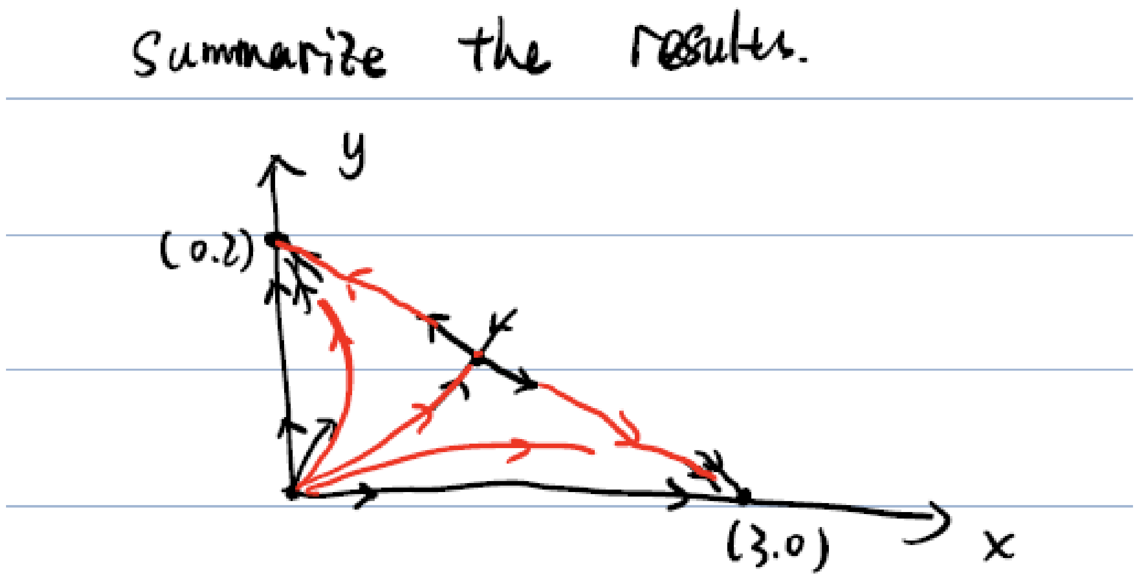

Figure 89: summarize fixed point population model

1. Stable manifold $\{x_0 : x(t) \to x^*, t \to \infty\}$. This is also called the basin of attraction (for the attracting fixed point). And the stable manifold of the saddle point gives the basin of boundary.

2. Unstable manifold $\{x_0 : x(t) \to x^*, t \to -\infty\}$

Combine with fixed points summary we get this figure (90):
Now we combine in addition the nullclines:

1. For $\dot{x} = 0$, we get $x = 0$ or $3 - x - 2y = 0$ so $y = \frac{3-x}{2}$

2. For $\dot{y} = 0$, we get $y = 0$ or $y = 2 - x$.

With all the values $\dot{x}$ and $\dot{y}$ (less than zero or bigger than zero) between regions, we get the figure (91) (drawing out the nullclines $y = 2 - x$ and $y = \frac{3-x}{2}$)

Example 181. consider the system

$$\begin{cases} \dot{x} &= y \\ \dot{y} &= x - x^3 \end{cases}$$

Then the energy $E = \frac{1}{2}y^2 - \frac{1}{2}x^2 + \frac{1}{4}x^4$. In this case we get a **homoclinic orbit:** a trajectory which connects a fixed point to itself. See figure (92)

Example 182. Consider the system:

$$\begin{cases} \dot{\theta} &= v \\ \dot{v} &= -\sin\theta \end{cases}$$

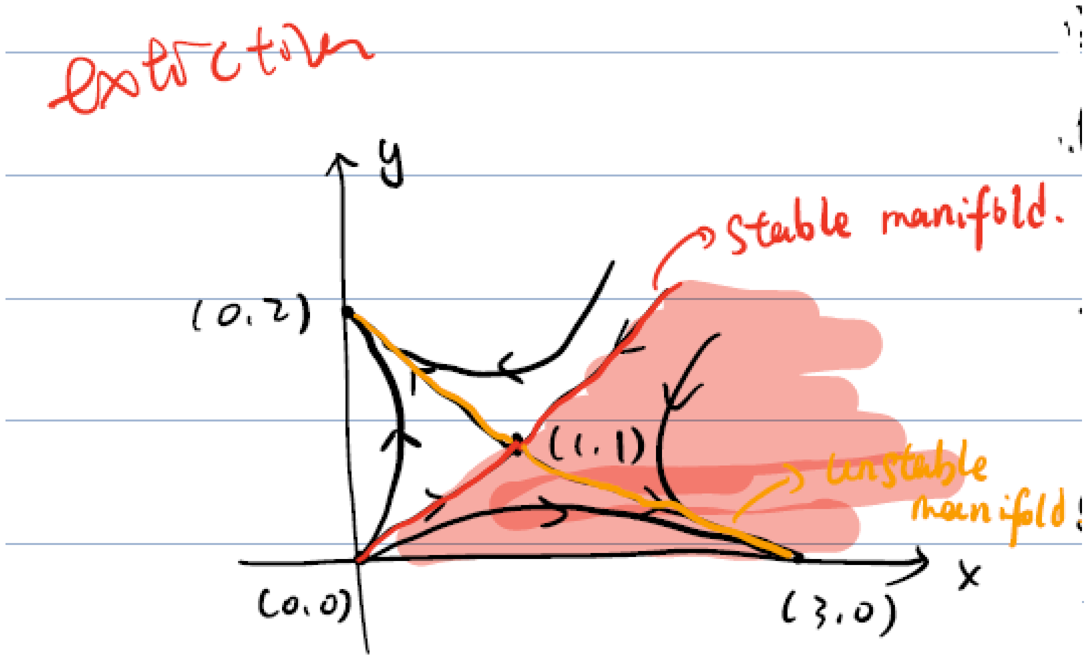

Figure 90: summarize fixed point with basin of attraction population model

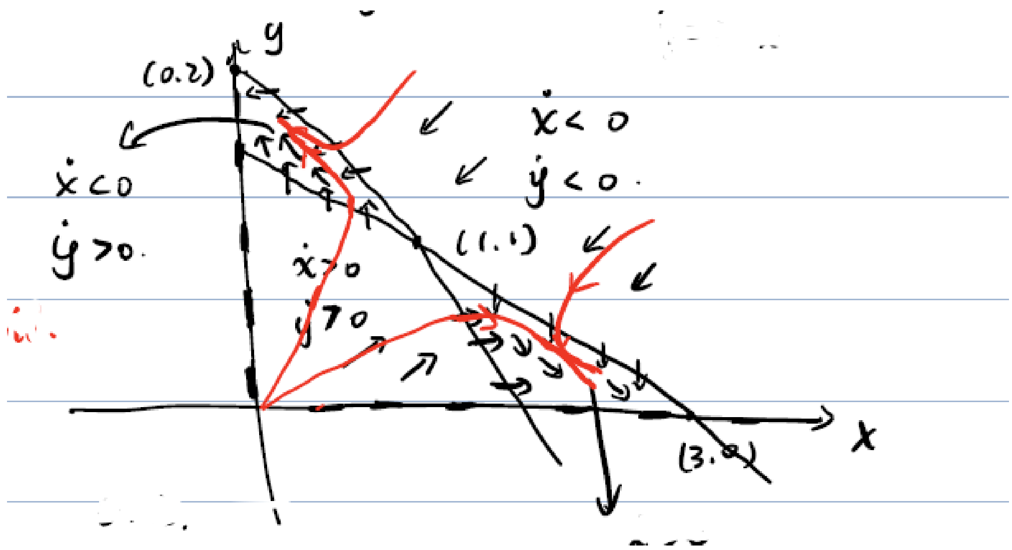

Figure 91: summarize fixed point with nullclines

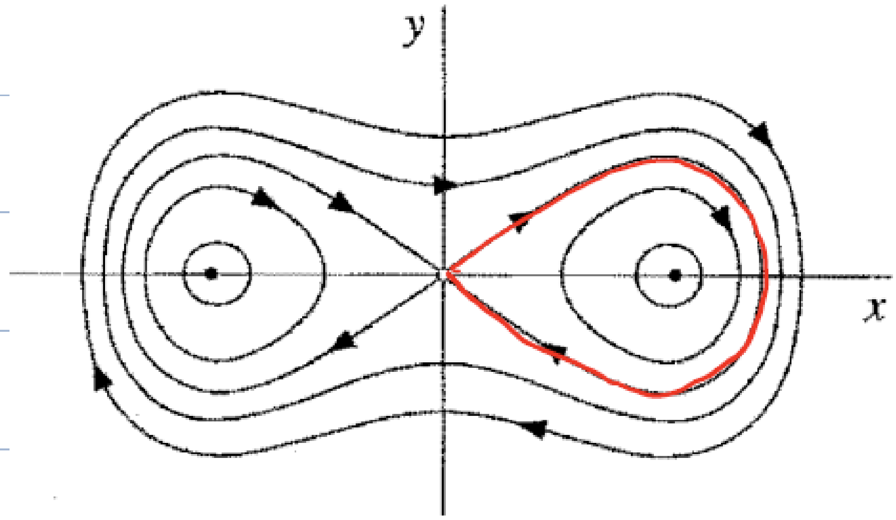

Figure 92: homoclinic orbit

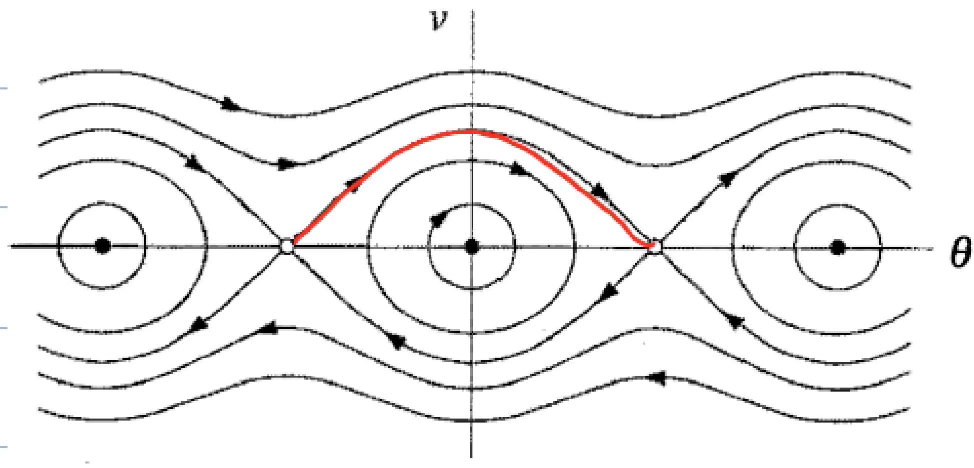

Figure 93: heteroclinic orbit

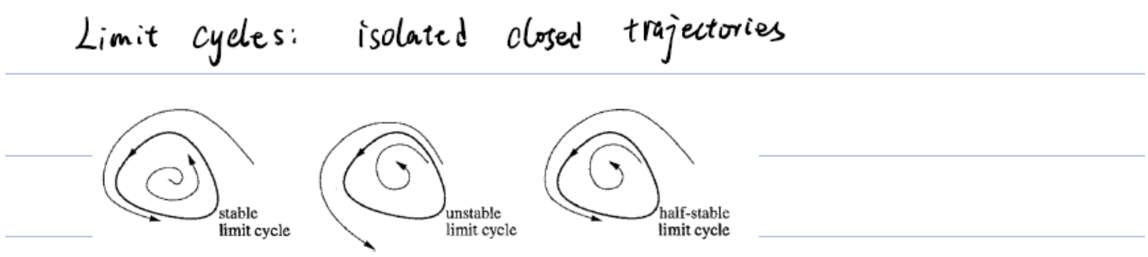

Figure 94: limit cycles

this is a pendulum. This is a **heteroclinic orbit:** a trajectory which connects distinct fixed points. See figure (93)

We have:

$$\begin{aligned}
\ddot{\theta} + \sin\theta &= 0 \\
\dot{\theta}(\ddot{\theta} + \sin\theta) &= 0 \\
\frac{1}{2}\dot{\theta}^2 - \cos\theta &= C = \text{ constant}
\end{aligned}$$

Because $E(\theta, v) = \frac{1}{2}v^2 - \cos\theta$. We also get the fixed point $(\theta^*, v^*) = (k\pi, 0)$.

3.5.1 Limit cycles: isolated closed trajectories

See figure (94) for pictures:

Example 183. Consider the system:

$$\begin{cases} \dot{x} &= x + y + \beta x(x^2 + y^2) \\ \dot{y} &= -x + y + \beta y(x^2 + y^2) \end{cases}$$

We get $x = r\cos\theta$ and $y = r\sin\theta$ when we convert it to polar coordinates, with $r^2 = x^2 + y^2$, $\theta = \arctan(\frac{y}{x})$. Therefore:

$$\begin{cases} \dot{r} &= r + \beta r^2 = r(1 + \beta r^2) \\ \dot{\theta} &= -1 \end{cases}$$

1. When $\beta \geq 0$: 1 fixed point $r = 0$. So the fixed point is the origin $(0, 0)$.

2. When $\beta < 0$: 2 fixed points on $r \geq 0$, $r^* = 0$ and $r^* = \sqrt{-\frac{1}{\beta}}$

See figure (95) for all the portraits.

Theorem 184 (Poincare-Bendixson Theorem)**.** *Suppose that:*

1. *D is a closed bounded subset of the plane $\mathbb{R}^2$.*

2. *$\overrightarrow{\dot{x}} = f(x)$ is a continuously differentiable vector field on an open set containing D.*

3. *D does not contain any fixed points.*

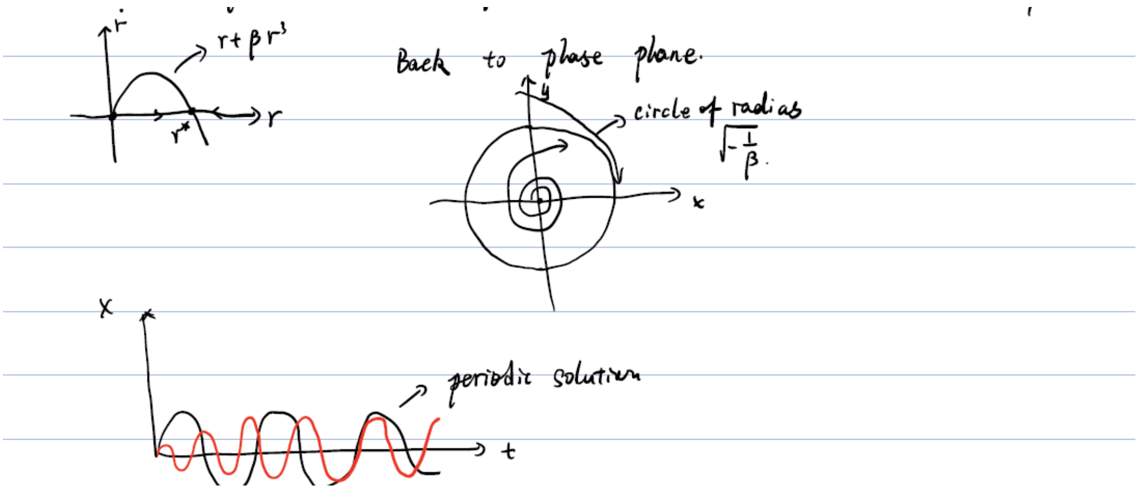

Figure 95: limit cycles example

4. *There exists a trajectory C that is confined in D, in the sense that it starts in D and stays for all future times.*

Then D contains a closed orbit.

To ensure the trajectory C exists, construct a trapping region Ω, i.e., a closed connected set such that the vector field points inward everywhere on the boundary of Ω. Then all trajectories in $\mathbb{R}$ is confined.

Example 185 (Glycolysis oscillations)**.** Consider the system

$$\begin{cases} \dot{x} &= -x + ay + x^2 y \\ \dot{y} &= b - ay - x^2 y \end{cases}$$

For $x, y \geq 0$ and $a, b > 0$.
The nullclines are:

1. When $\dot{x} = 0$:

$$-x + ay + x^y y = 0$$
$$y = \frac{x}{a + x^2}$$

2. When $\dot{y} = 0$:

$$b - ay - x^2 y = 0$$
$$y = \frac{b}{a + x^2}$$

See figure (96) for nullclines diagram:

Now we try to construct that region of closed orbit D. We just need to show that:

$$(-\dot{y}) - \dot{x} > 0$$

The constructed region is a trapping region.
The fixed point of the system is $x^* = b$ and $y^* = \frac{b}{a+b^2}$. See figure (97) for the construction process:

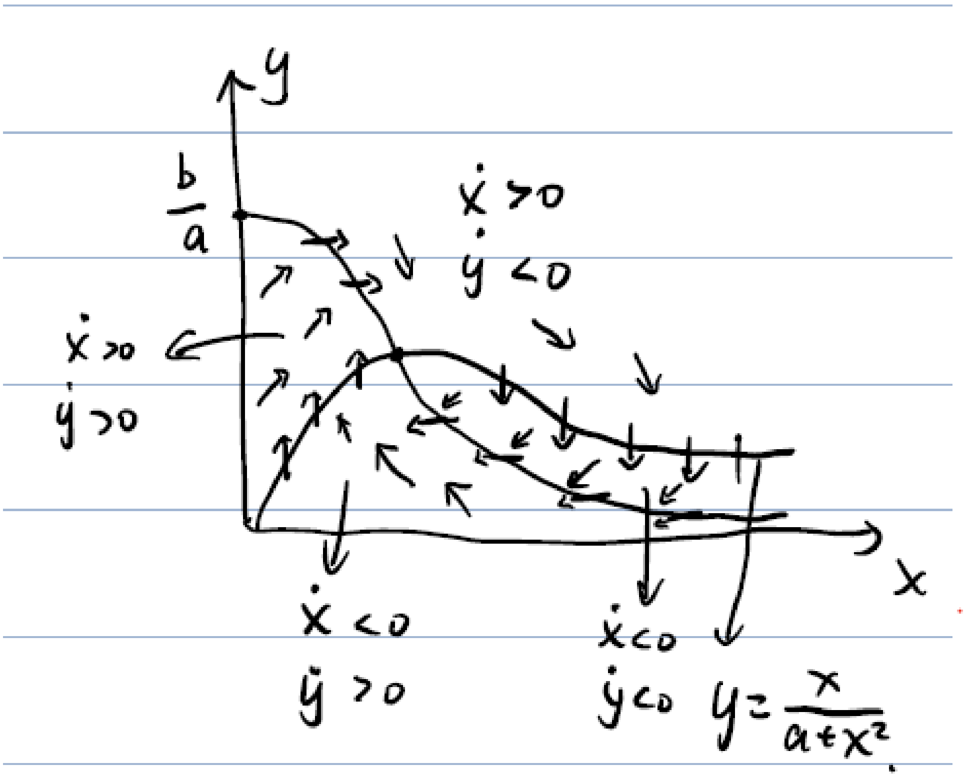

Figure 96: Glycolysis oscillations nullclines

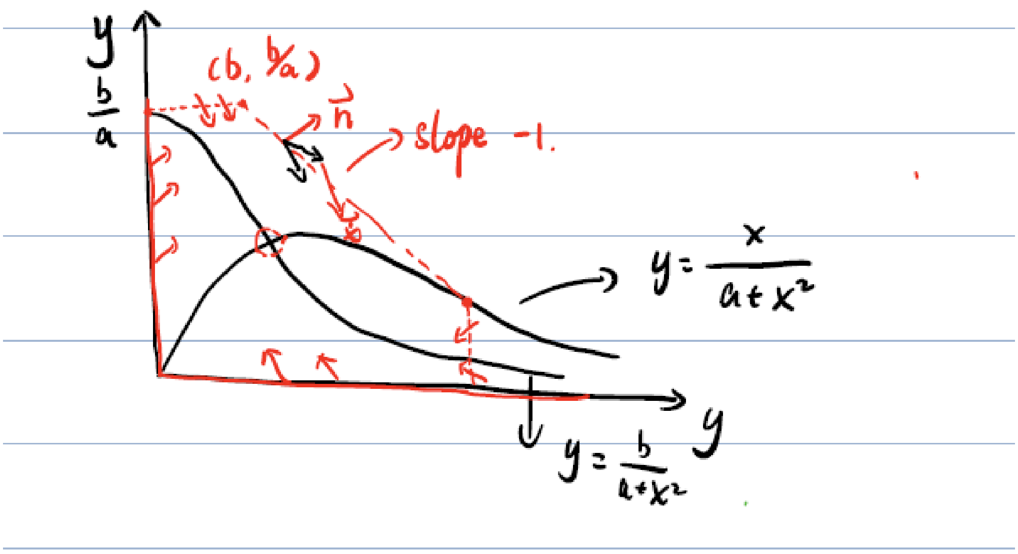

Figure 97: Glycolysis oscillations closed orbit construction

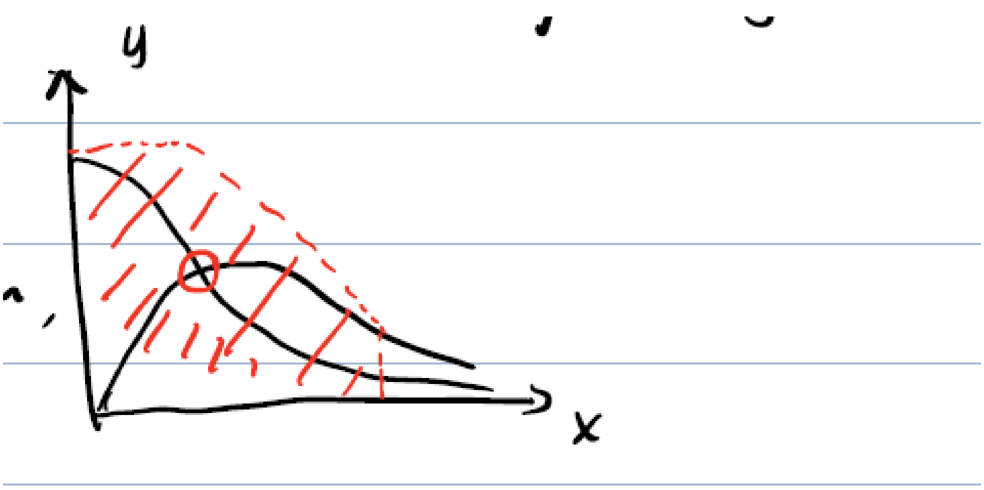

Figure 98: glycolysis oscillations region D by Poincare Bendixson theorem

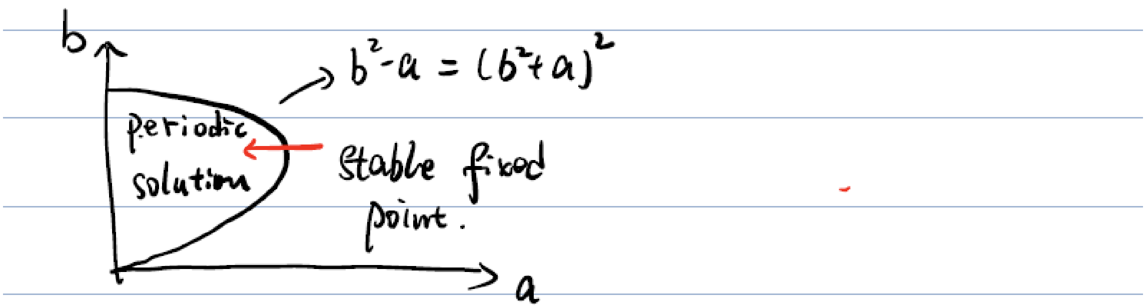

Figure 99: glycolysis oscillations based on a and b

The Jacobian matrix is:

$$J = \begin{pmatrix} -1+2xy & a+x^2 \\ -2xy & -(a+x^2) \end{pmatrix} \Bigg|_{x^*,y^*} = \begin{pmatrix} \frac{b^2-a}{b^2+a} & b^2+a \\ \frac{-2b^2}{a+b^2} & -(a+b^2) \end{pmatrix}$$

Then $\det J = a + b^2 > 0$ since $a, b > 0$, and $\mathrm{tr}J = \frac{(b^2-a)-(b^2+a)^2}{b+a}$. The fixed point is unstable for $\mathrm{tr}J > 0$ and stable for $\mathrm{tr}J < 0$. Then in this case, when $\mathrm{tr}J > 0$, consider the region by the Poincare-Bendixson theorem, there is a limit cycle in the given region. See the figure (98) for this region.

In addition, depends on a, b we have this figure (99)

3.6 Day 19: More on Population problem

3.6.1 Bifurcation

A revisit to saddle nodes, transcritical, and pitchfork.

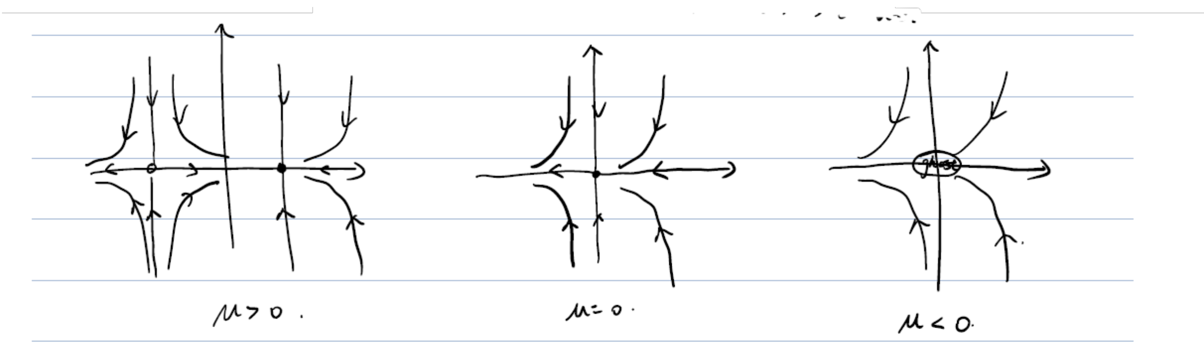

Figure 100: general bifurcation in population example

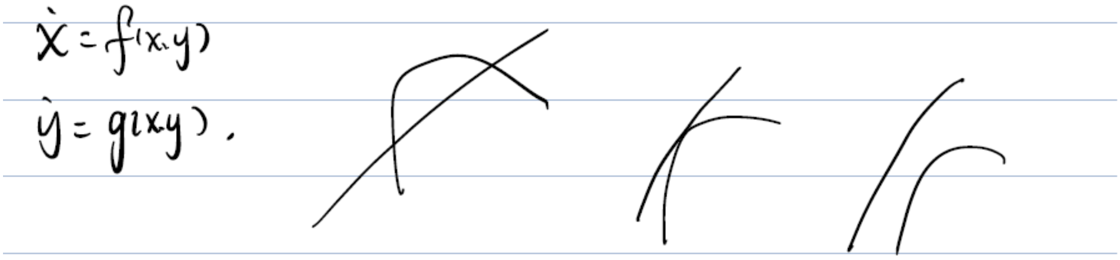

Figure 101: sketch of bifurcation

Example 186. Consider the system

$$\begin{cases} \dot{x} &= \mu - x^2 \\ \dot{y} &= -y \end{cases}$$

The fixed points are $(\pm\sqrt{\mu}, 0)$. The Jacobian matrix is:

$$J = \begin{pmatrix} -2x & 0 \\ 0 & -1 \end{pmatrix}$$

We see that $(-\sqrt{\mu}, 0)$ saddle and $(\sqrt{\mu}, 0)$ stable. The bifrucation diagram is in figure (100).

Generally, the system:

$$\begin{cases} \dot{x} &= f(x,y) \\ \dot{y} &= g(x,y) \end{cases}$$

has the graph like figure (101)

The bifurcations are:

1. Transcritical bifurcation: $\dot{x} = \mu x - x^2$ and $\dot{y} = -y$.

2. Pitchfork bifurcation: $\dot{x} = \mu x \pm x^3$ and $\dot{y} = -y$.

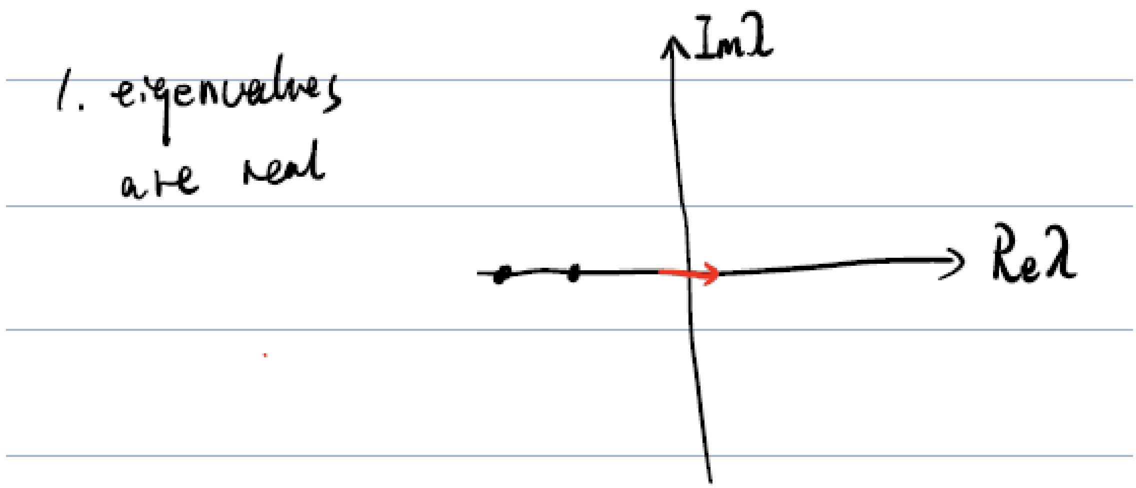

Figure 102: Hopf bifurcation real eigenvalues

3.6.2 Hopf bifurcation

The system is

$$\frac{d\overrightarrow{x}}{dt} = \overrightarrow{F}(\overrightarrow{x}, a) \in \mathbb{R}^2$$

Suppose there is a fixed point $\overrightarrow{x_0}$. That is $\overrightarrow{F}(x_0, a) = 0$ where $\overrightarrow{x_0} = \overrightarrow{x_0}(a)$.
Suppose that the fixed point is stable, and it becomes unstable if it varies. What happens to the eigenvalues of the Jacobian matrix?

1. Eigenvalues are real (see figure(102))
2. Eigenvalues are complex (see figure (103))

But we could have a new type of bifurcation: **Hopf bifurcation**.

Example 187. For example, consider

$$J = \begin{pmatrix} \mu & \nu \\ -\nu & \mu \end{pmatrix}$$

With $\lambda = \mu \pm i\nu$. Suppose $\mu(a_0) = 0$ and $\mu'(a_0) > 0$. The Hopf bifurcation happens at $a = a_0$. See figure (104).

Two types of Hopf bifurcation are supercritical and subcritical in figure (105):

Example 188. Glycolysis: When $\text{tr}J = 0$ and $\det J > 0$, Hopf bifurcation happens.

4 Past Qualifying Exams

4.1 Fall 2021

Answers 3 of the following 5 problems:

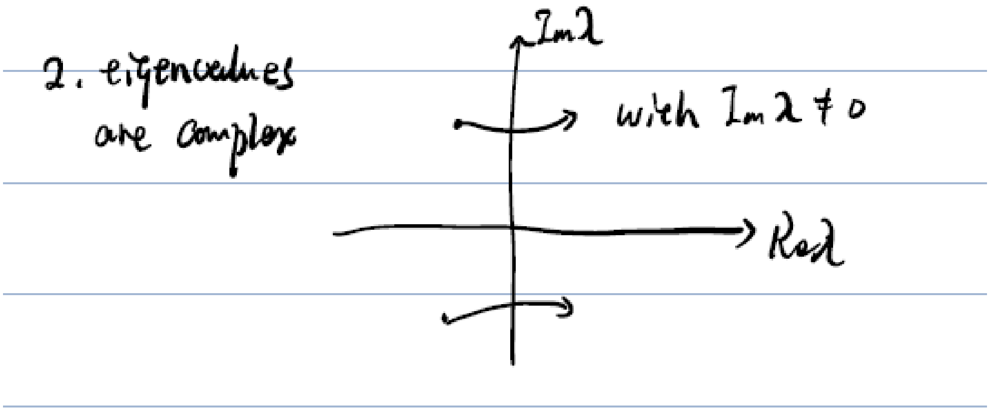

Figure 103: bifurcation complex eigenvalues

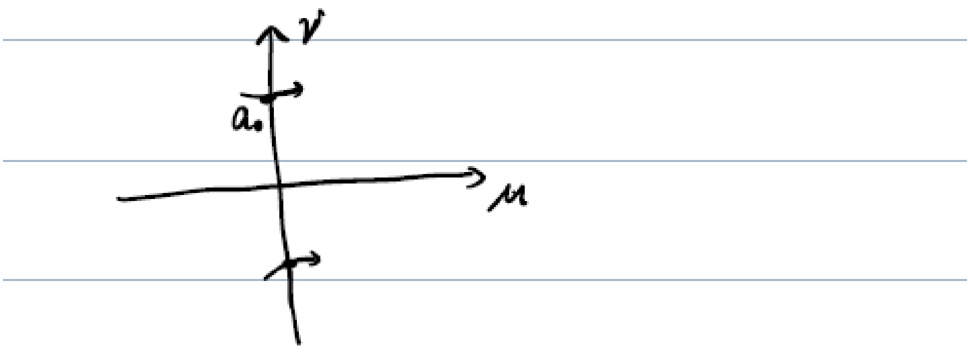

Figure 104: Hopf bifurcation happens at a_0

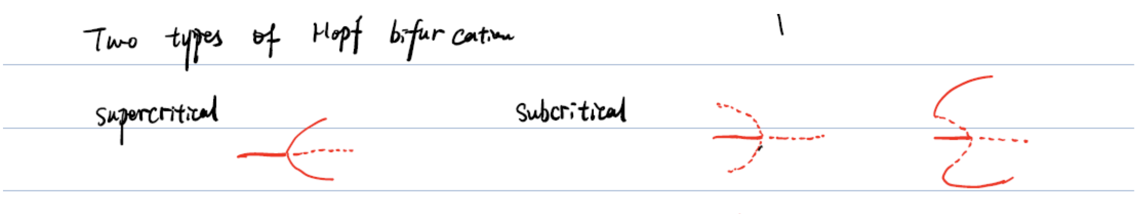

Figure 105: two types of Hopf bifurcation

Problem 1.

1. (2 points) Describe the fundamental lemma of the calculus of variations.
2. (3 points) Describe the Brachistochrone problem and derive the functional for it when initial velocity is zero, Do not solve the variational problem.
3. (5 points) Show that if $y(x)$ is an extremum of the functional $J[y(x)] = \int_a^b F(x, y, y')dx$, and if the endpoints $y(a)$ and $y(b)$ are not specified, then $y(x)$ must satisfy the Euler-Lagrange Equation and the natural boundary conditions.

Hints. Should?

Proof. How $\square$

Problem 2.

1. (2 points) Describe the definition of the action function.
2. (3 points) Define the canonical variable, and write down the Legendre transform and the canonical form of the Euler-Lagrange equation.
3. (5 points) Find the extremal of the functional $J[y] = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + (r')^2} d\theta$ where $r = r(\theta)$.

Hints. Yay?

Proof. Not... $\square$

Problem 3.

1. (3 points) Explain the connection of the solution of the Euler-Lagrange equation with solution of the hamilton-Jacobi equation.
2. (4 points) Given the functional $J[y(x)] = \int_{x_0}^{x_1} F(x, y, y')dx$. Prove that the transversality is the same as orthogonality if and only if $F(x, y, y')$ has the form $F(x, y, y') = g(x, y)\sqrt{1 + (y')^2}$ with $g(x, y) \neq 0$ near the point of intersection.
3. (3 points) Give the definition of the fixed points of continuous and discrete dynamical systems, and describe how to determine whether the fixed points are hyperbolic or not.

Hints. Nah

Proof. Well $\square$

Problem 4.

1. (2 points) Describe the definition of topological conjugacy.
2. (2 points) Give a definition of a chaotic discrete dynamic system
3. (4 points) Determine the stability of fixed points and the period-2 cycle of the discrete dynamical system $x_{n+1} = rx_n(1 - x_n)$ when $r \in (3, 1 + \sqrt{6}$.

4. (2 points) Determine the first four elements in the itinerary with $x_0 = 1/8$ for the logistic map with $r = 4$

Hints. Could be

Proof. Nah ☐

Problem 5.

1. (2 points) For a two-dimensional continuous system, give the definition of nullclines, and describe how it can help plot the phase portrait of the system.
2. (4 points) Consider the dynamic system $\dot{x} = rx + 4x^3$, find the bifurication points and identify the type of bifurication.
3. (4 points) For the following system:

$$\begin{aligned} \dot{x} &= x(3 - x - y) \\ \dot{y} &= y(2 - x - y) \end{aligned}$$

   find and classify all of the fixed points. Determine whether the relation between the 2 species x and y is competition or cooperation, and explain.

Hints. Nah

Proof. yes ☐