Math 46 - Fall 2021 - Discussion section 003 and 005
Instructor: Prof. Feng Xu
Summary of some methods to solve ODE

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Contents

1 Disclaimer

This document is prepared in helping my students with their class. It certainly contain errors, and if you spot any error or have any comment about it, please do not hesititate to contact me at my email **khoi.vo@email.ucr.edu**.

Happy Learning together!!

Khoi.

2 Linear First Order Equations

If we have a form $y' + p(x)y = 0$ where $p(x)$ is a function of x only, this is called the **homogeneous** (linear first order equation).

Example 1. Here is an example of a homogeneous (linear first order) ODE:

$$y' + \frac{1}{x}y = 0$$

In this example, $p(x) = \frac{1}{x}$.

If we did not get a 0 on the right hand side, but instead get a function of x, for instance, $f(x)$, and the ODE looks something like $y' + p(x)y = f(x)$, this is called **non-homogeneous**.

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Example 2. Here is an example of a non-homogeneous (linear first order) ODE:

$$y' + \frac{1}{x}y = x^2$$

2.1 Homogeneous Linear First order

We will now summarize steps to solve for a homogeneous linear first order ODE. Let's use an example and try to solve that example:

Example 3. 1. Find a general solution y to this ODE:

$$y' + \frac{1}{x^2}y = 0$$

In this example, $p(x) = \frac{1}{x^2}$.

2. With the initial condition $y(1) = 1$, find the particular solution.

Solution. The textbook might do something such as the integral factor $I(x) = \int p(x)dx$. And in this case we can get the integration factor is $I(x) = \int \frac{1}{x} = \ln|x| + D$. Then the general solution will be $y = ce^{-I}$, which is

$$y = ce^{-I} = ce^{-(\ln|x|+D)} = ce^{x^{-1}} = ce^{-D}e^{-\ln|x|} = Ke^{-1}.$$

in which K is the constant (could be positive or negative) ce^{-D}

But I do not recommend doing this way, unless you can memorize the integration factor formula, and the general solution will be a constant multiply with e to the power of negative of that integration factor.

Step 1: Here is a way I usually do it with less memorization: **"Separate" x on one side and all the other y', y on the other side** to get:

$$\frac{y'}{y} = -\frac{1}{x}.$$

Step 2: Then you do integration, on the left hand side (everything is in y) so you do with respect to y, while on the left integrate with respect to x:

$$\ln(|y|) = -\ln|x| + C$$

Step 3: Now this has some technicality $|y|$ instead of y, but because we are raising exponent e of both sides, says:

$$|y| = e^{-\ln|x|+C} = e^C e^{\ln(|x|^{-1})} = e^C|x|^{-1}$$

We can think of e^C as some constant which could be positive or negative and write it as a K so that we can break the absolute values of y and x and get:

$$y = Kx^{-1}$$

And note that this is exactly the same as when we do the integration factor I.

Step 4: Solve for the constant K by the initial condition, that is, when $x = 1$, $y = 1$:

$$1 = K(1) = K$$

And we get the answer as

$$y = x^{-1}$$

2.2 Non-homogeneous Linear First Order

Let's use the earlier example:

Example 4. 1. Find a general solution y to this ODE:

$$y' + \frac{1}{x}y = x^2$$

2. With the initial condition $y(1) = 1$, find the particular solution.

Solution. I would like to introduce you with this idea below:

Step 1: This is an important idea in ODE, which is, to **start with the homogeneous case** and then build your actual solution of the non-homogeneous case based on the answer from the homogeneous case. So from the example in the previous part we get $y_1 = x^{-1}$ is a solution to the homogeneous case. Note that we do **NOT** include the constant K, as we will build our new solution with an unknown factor called $u(x)$ which might also have constants and variable x in it. We do not want to make the situation more complicated.

Step 2: Set

$$y = u(x)y_1 = u(x)x^{-1}$$

as a "guessed" solution to the non-homogeneous equation. Our job now will be find this $u(x)$. In words, we "scaled" the homogeneous solutions y_1 by a "factor" $u(x)$.

Step 3 : To find $u(x)$, we need to plug y back into the non-homogeneous equation. And first we need to find y'. Here, remember to use the **product rule** because y is the product of two functions of x the $u(x)$ and the x^{-1}.

$$y' = u'x^{-1} - ux^{-2}$$

Then we plug it into the non-homogeneous function Left Hand Side:

$$y' + \frac{1}{x}y = u'x^{-1} - ux^{-2} + \frac{1}{x}ux^{-1} = u'x^{-1}$$

since $\frac{1}{x}ux^{-1} = ux^{-2}$ so the last 2 terms canceled each other. Then compare this with the RHS (right hand side) of the non-homogeneous equation and we get:

$$u'x^{-1} = x^2$$

So

$$u'(x) = x^{-3}$$

Then we integration with respect to x and we can find $u(x) = \frac{x^{-2}}{-2} + C = \frac{1}{-2x^2} + C$

Step 4: Find the constant C:

$$y(x) = u(x)x^{-1} = (\frac{1}{-2x^2} + C)x^{-1}$$

When $x = 1$ then $y = 1$ gives:

$$1 = (\frac{1}{-2} + C)(1)$$

So $C = \frac{3}{2}$. Hence our particular solution to the non-homogeneous equation is

$$y = (\frac{1}{-2x^2} + \frac{3}{2})x^{-1} = \frac{1}{-2x^3} + \frac{3}{2x}$$

3 Separable

As the name said it all, you can usually separate the x and dx on one side and the other side will have only the y and dy. Then you just integrate each side correspondingly (wrt x on the side of dx and wrt y on the side of dy.

Example 5. Solve the equation $y' = x(1 + y^2)$

Solution. Rewrite the original equation a little bit into

$$\frac{y'}{1 + y^2} = x$$

Or

$$\frac{1dy}{1 + y^2} = xdx$$

Then integrate both sides:

$$\arctan(y) = \frac{x^2}{2} + C$$

And taking the tan give:

$$y = \tan(\frac{x^2}{2} + C)$$