Math 31 - Fall 2021 - Discussion 10

1. (a) What is the identity matrix of size $n \times n$? What do we get when we multiply the identity matrix by another matrix of compatible size?
   (b) What does it mean for an $n \times n$ matrix A to be invertible?
   (c) What is the inverse of the product ABC where A, B, C are invertible $n \times n$ matrices?
   (d) What is the criterion in terms of the numbers a, b, c, d to know that the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

   is invertible?

2. Find the inverse of each of the matrices

$$A = \begin{pmatrix} 2 & -3 \\ -5 & 4 \end{pmatrix}, \qquad B = \begin{pmatrix} 3 & 4 \\ -2 & -3 \end{pmatrix}.$$

3. (a) The columns of every invertible $n \times n$ matrix are linearly independent. Explain why.
   (b) The columns of any invertible $n \times n$ matrix span $\mathbb{R}^n$. Explain why.

4. Determine if the following matrix is invertible and if it is invertible find its inverse,

$$A = \begin{pmatrix} 1 & -2 & 1 \\ -2 & 5 & -3 \\ 4 & -6 & 2 \end{pmatrix}.$$

5. Find the inverse of the matrix

$$A = \begin{pmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{pmatrix}.$$

   You should get

$$A^{-1} = \begin{pmatrix} 8 & 3 & 1 \\ 10 & 4 & 1 \\ \frac{7}{2} & \frac{3}{2} & \frac{1}{2} \end{pmatrix}.$$

   Use this to solve the system of equations $Ax = b$ where

$$b = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}.$$

1/ a/

$$\begin{bmatrix} 1 & 0 & 0 & & & \\ 0 & 1 & 0 & 0 & 0 & \\ 0 & 0 & 1 & 0 & 0 & \\ 0 & 0 & 0 & 1 & 0 & \\ \cdots & & & & & \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \quad O$$

n rows

n columns

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$I \cdot A = A$$

b/ invertible: A is invertible

there is B special

$$\begin{cases} 1/\ B \cdot A = I \\ 2/\ A \cdot B = I \end{cases}$$ identity matrix

denote $B = A^{-1}$

$$A = \begin{bmatrix} 2 & 1 & 0 \\ \cdot & - & - \\ \cdot & \cdot & \cdot \end{bmatrix}$$

$A^{-1} = [$ ~~1/2~~ ?? ???

d/ $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = A$

$$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

only possible when $ad - bc \neq 0$.

c/ $(ABC)^{-1} = C^{-1} B^{-1} A^{-1}$

c*/ $(ABCD)^{-1} = D^{-1} C^{-1} B^{-1} A^{-1}$

Math 31 - Fall 2021 - Discussion 10

1. (a) What is the identity matrix of size $n \times n$? What do we get when we multiply the identity matrix by another matrix of compatible size?
   (b) What does it mean for an $n \times n$ matrix A to be invertible?
   (c) What is the inverse of the product ABC where A, B, C are invertible $n \times n$ matrices?
   (d) What is the criterion in terms of the numbers a, b, c, d to know that the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

   is invertible?

2. Find the inverse of each of the matrices

$$A = \begin{pmatrix} 2 & -3 \\ -5 & 4 \end{pmatrix}, \qquad B = \begin{pmatrix} 3 & 4 \\ -2 & -3 \end{pmatrix}.$$

3. (a) The columns of every invertible $n \times n$ matrix are linearly independent. Explain why.
   (b) The columns of any invertible $n \times n$ matrix span $\mathbb{R}^n$. Explain why.

4. Determine if the following matrix is invertible and if it is invertible find its inverse,

$$A = \begin{pmatrix} 1 & -2 & 1 \\ -2 & 5 & -3 \\ 4 & -6 & 2 \end{pmatrix}.$$

5. Find the inverse of the matrix

$$A = \begin{pmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{pmatrix}.$$

   You should get

$$A^{-1} = \begin{pmatrix} 8 & 3 & 1 \\ 10 & 4 & 1 \\ \frac{7}{2} & \frac{3}{2} & \frac{1}{2} \end{pmatrix}.$$

   Use this to solve the system of equations $Ax = b$ where

$$b = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}.$$

2/a/ $A = \begin{pmatrix} 2 & -3 \\ -5 & 4 \end{pmatrix}$

$$A^{-1} = \frac{1}{2\cdot 4 - (-5)(-3)} \begin{pmatrix} 4 & 3 \\ 5 & 2 \end{pmatrix}$$

$$= \frac{1}{-7} \begin{pmatrix} 4 & 3 \\ 5 & 2 \end{pmatrix} = \begin{pmatrix} -\frac{4}{7} & -\frac{3}{7} \\ -\frac{5}{7} & -\frac{2}{7} \end{pmatrix}$$

b/ $B = \begin{pmatrix} 3 & 4 \\ -2 & -3 \end{pmatrix}$

$$B^{-1} = \frac{1}{3(-3) - (-2)(4)} \begin{pmatrix} -3 & -4 \\ 2 & 3 \end{pmatrix}$$

$$= \frac{1}{-1} \begin{pmatrix} -3 & -4 \\ 2 & 3 \end{pmatrix}$$

$$= -1 \begin{pmatrix} -3 & -4 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ -2 & -3 \end{pmatrix} = B.$$

$B^{-1} = B$: $\quad B^{-1} \cdot B = \begin{pmatrix} 3 & 4 \\ -2 & -3 \end{pmatrix} \begin{pmatrix} 3 & 4 \\ -2 & -3 \end{pmatrix}$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I.$$

Math 31 - Fall 2021 - Discussion 10

1. (a) What is the identity matrix of size $n \times n$? What do we get when we multiply the identity matrix by another matrix of compatible size?
   (b) What does it mean for an $n \times n$ matrix A to be invertible?
   (c) What is the inverse of the product ABC where A, B, C are invertible $n \times n$ matrices?
   (d) What is the criterion in terms of the numbers a, b, c, d to know that the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

   is invertible?

2. Find the inverse of each of the matrices

$$A = \begin{pmatrix} 2 & -3 \\ -5 & 4 \end{pmatrix}, \qquad B = \begin{pmatrix} 3 & 4 \\ -2 & -3 \end{pmatrix}.$$

3. (a) The columns of every invertible $n \times n$ matrix are linearly independent. Explain why.
   (b) The columns of any invertible $n \times n$ matrix span $\mathbb{R}^n$. Explain why.

4. Determine if the following matrix is invertible and if it is invertible find its inverse,

$$A = \begin{pmatrix} 1 & -2 & 1 \\ -2 & 5 & -3 \\ 4 & -6 & 2 \end{pmatrix}.$$

5. Find the inverse of the matrix

$$A = \begin{pmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{pmatrix}.$$

   You should get

$$A^{-1} = \begin{pmatrix} 8 & 3 & 1 \\ 10 & 4 & 1 \\ \frac{7}{2} & \frac{3}{2} & \frac{1}{2} \end{pmatrix}.$$

   Use this to solve the system of equations $Ax = b$ where

$$b = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}.$$

3/ $A x = \vec{0}$

then $x_1 = x_2 = x_3 = \dots = x_n = 0$.

$$\vec{x} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} = \vec{0}$$

$$\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} + x_3 \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

should get : $x_1 = x_2 = x_3 = 0$

A invertible means there is A^{-1} so that:

$$A^{-1} A = A A^{-1} = I$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$(A\vec{x}) = \vec{0}$$

$\downarrow A^{-1}$

$$A^{-1}(A\vec{x}) = A^{-1}\vec{0}$$

$$(A^{-1}A)\vec{x} = 0$$

$$I\vec{x} = 0 \quad \nearrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{x} = 0$$

Math 31 - Fall 2021 - Discussion 10

1. (a) What is the identity matrix of size $n \times n$? What do we get when we multiply the identity matrix by another matrix of compatible size?
   (b) What does it mean for an $n \times n$ matrix A to be invertible?
   (c) What is the inverse of the product ABC where A, B, C are invertible $n \times n$ matrices?
   (d) What is the criterion in terms of the numbers a, b, c, d to know that the matrix
   $$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$
   is invertible?

2. Find the inverse of each of the matrices
$$A = \begin{pmatrix} 2 & -3 \\ -5 & 4 \end{pmatrix}, \qquad B = \begin{pmatrix} 3 & 4 \\ -2 & -3 \end{pmatrix}.$$

3. (a) The columns of every invertible $n \times n$ matrix are linearly independent. Explain why.
   (b) The columns of any invertible $n \times n$ matrix span $\mathbb{R}^n$. Explain why.

4. Determine if the following matrix is invertible and if it is invertible find its inverse,
$$A = \begin{pmatrix} 1 & -2 & 1 \\ -2 & 5 & -3 \\ 4 & -6 & 2 \end{pmatrix}.$$

5. Find the inverse of the matrix
$$A = \begin{pmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{pmatrix}.$$
You should get
$$A^{-1} = \begin{pmatrix} 8 & 3 & 1 \\ 10 & 4 & 1 \\ \frac{7}{2} & \frac{3}{2} & \frac{1}{2} \end{pmatrix}.$$
Use this to solve the system of equations $Ax = b$ where
$$b = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}.$$

Theorem 4, p. 37: (a) and (e)

column of n by n span $\mathbb{R}^n$ if

for each b in $\mathbb{R}^n$: $Ax = b$ has a unique solution.

A is invertible means there is A^{-1} that:

$$A^{-1}A = AA^{-1} = I$$

$$A\vec{x} = \vec{b}$$

(multiply with A^{-1})

$$A^{-1}(A\vec{x}) = A^{-1}\vec{b}$$

$$(A^{-1}A)\vec{x} = A^{-1}\vec{b}$$

$$I\vec{x} = A^{-1}\vec{b}$$

$$\vec{x} = A^{-1}\vec{b} \text{ : solution}$$

A^{-1}: only 1

$\vec{b}$: only 1

⇓

$\vec{x}$ only 1

$\vec{x}$ unique.

Math 31 - Fall 2021 - Discussion 10

1. (a) What is the identity matrix of size $n \times n$? What do we get when we multiply the identity matrix by another matrix of compatible size?

   (b) What does it mean for an $n \times n$ matrix A to be invertible?

   (c) What is the inverse of the product ABC where A, B, C are invertible $n \times n$ matrices?

   (d) What is the criterion in terms of the numbers a, b, c, d to know that the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

   is invertible?

2. Find the inverse of each of the matrices

$$A = \begin{pmatrix} 2 & -3 \\ -5 & 4 \end{pmatrix}, \qquad B = \begin{pmatrix} 3 & 4 \\ -2 & -3 \end{pmatrix}.$$

3. (a) The columns of every invertible $n \times n$ matrix are linearly independent. Explain why.

   (b) The columns of any invertible $n \times n$ matrix span $\mathbb{R}^n$. Explain why.

4. Determine if the following matrix is invertible and if it is invertible find its inverse,

$$A = \begin{pmatrix} 1 & -2 & 1 \\ -2 & 5 & -3 \\ 4 & -6 & 2 \end{pmatrix}.$$

5. Find the inverse of the matrix

$$A = \begin{pmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{pmatrix}.$$

   You should get

$$A^{-1} = \begin{pmatrix} 8 & 3 & 1 \\ 10 & 4 & 1 \\ \frac{7}{2} & \frac{3}{2} & \frac{1}{2} \end{pmatrix}.$$

   Use this to solve the system of equations $Ax = b$ where

$$b = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}.$$

Theorem 8: (p. 114)

A has pivot position exactly n.

$$\begin{pmatrix} 1 & -2 & 1 \\ -2 & 5 & -3 \\ 4 & -6 & 2 \end{pmatrix}$$

$2R_1 + R_2 \to R_2$

$-4R_1 + R_3 \to R_3$

$$\begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 2 & -2 \end{pmatrix}$$

$$A \quad \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$n=3$: Do I have 3 pivot?

NO :(b/c last row is all 0s.

A is NOT invertible

Theorem 8 part c