Math 31 - Fall 2021 - Week 3 - Discussion 7

1. Let $u = \binom{3}{5}$, $b = \begin{pmatrix} 4 \\ 16 \\ 2 \end{pmatrix}$, and

   $\mathbb{R}^2$ $\mathbb{R}^3$ $\mathbb{R}^2$

   $$\boxed{A = \begin{pmatrix} -2 & 0 \\ 0 & 4 \\ 1 & 1 \end{pmatrix}}. \qquad A x$$

   Define a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ by $T(x) = Ax$.
   Find $T(u)$, the image of u. Find the solutions of $T(x) = b$, the preimage of b.

2. Describe what the following matrix does, as a linear transformation.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

3. Let

$$A = \begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix}.$$

   Draw a square from $(0, 0)$ to $(2, 2)$. Determine how $T(x) = Ax$ acts on the square. This is called a *shear transformation.*

4. Give an example of a map $f : \mathbb{R} \to \mathbb{R}$ which is *not* linear. What property does it not satisfy?

5. Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(x) = -5x$. Show that T is a linear transformation.

6. Consider the matrix

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

   Define $T(x) = Ax$, and find the image of $u = \binom{5}{2}$, $v = \binom{1}{6}$, and $u + v$.

1/ $T(u) = Au =$

$$\begin{pmatrix} -2 & 0 \\ 0 & 4 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 5 \end{pmatrix} = \begin{pmatrix} -6 \\ 20 \\ 8 \end{pmatrix}$$

$$\underbrace{3 \text{ by } 2 \quad 2 \text{ by } 1}_{3 \text{ by } 1}$$

$$\begin{pmatrix} -2 & 0 \\ 0 & 4 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 4 \\ 16 \\ 2 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} -2 & 0 & 4 \\ 0 & 4 & 16 \\ 1 & 1 & 2 \end{array}\right)$$

$(R_2 \times 2 + R_1 \leftrightarrow R_2$

$$\left(\begin{array}{cc|c} -2 & 0 & 4 \\ 0 & 4 & 16 \\ 0 & 2 & 8 \end{array}\right)$$

$4x_2 = 16$

$2x_2 = 8$

$$\boxed{x_2 = 4}$$

$-2x_1 = 4 \rightarrow \boxed{x_1 = -2}$

$\boxed{1 \quad 0 \quad -2}$ $\quad x_1 = -2$

$$\begin{pmatrix} -2 \\ 4 \end{pmatrix}$$

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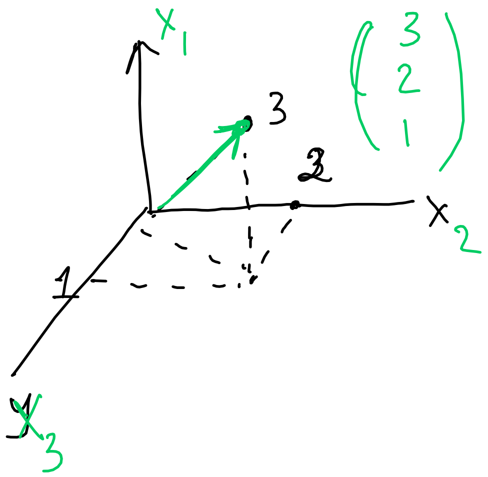
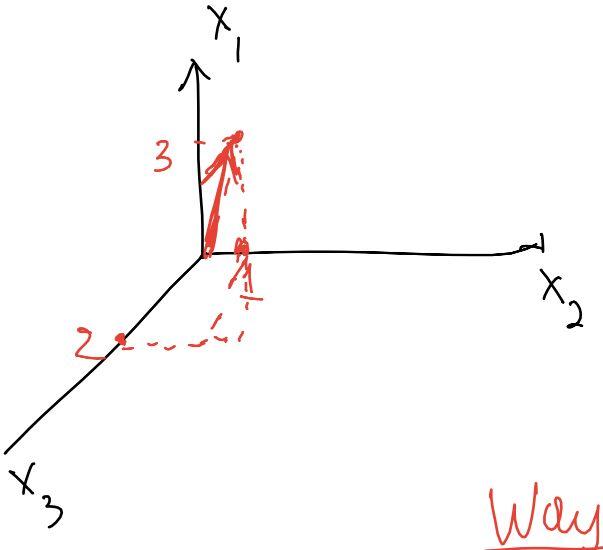
$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 5 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$$

2/ $\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}x_1\\x_3\\x_2\end{pmatrix}$

3 by 3 $\quad$ 3 by 1

$$\begin{pmatrix}x_1\\ \boxed{\begin{matrix}x_2\\x_3\end{matrix}}\end{pmatrix}\longmapsto\begin{pmatrix}x_1\\ \boxed{\begin{matrix}x_3\\x_2\end{matrix}}\end{pmatrix}$$





6/ $A=\begin{pmatrix}0&-1\\1&0\end{pmatrix}$

$$Tu=Au=\begin{pmatrix}0&-1\\1&0\end{pmatrix}\begin{pmatrix}5\\2\end{pmatrix}=\boxed{\begin{pmatrix}-2\\5\end{pmatrix}}$$

$$Tv=Av=\begin{pmatrix}0&-1\\1&0\end{pmatrix}\begin{pmatrix}1\\6\end{pmatrix}=\boxed{\begin{pmatrix}-6\\1\end{pmatrix}}$$

$$T(u+v)=T(u)+T(v)$$

Way 2

$$=\begin{pmatrix}-2\\5\end{pmatrix}+\begin{pmatrix}-6\\1\end{pmatrix}=\begin{pmatrix}-8\\6\end{pmatrix}\longleftrightarrow=\begin{pmatrix}-8\\6\end{pmatrix}$$

Way 1:

$$u+v=\begin{pmatrix}5\\2\end{pmatrix}+\begin{pmatrix}1\\6\end{pmatrix}=\begin{pmatrix}6\\8\end{pmatrix}$$

$$T(u+v)=A(u+v)=\begin{pmatrix}0&-1\\1&0\end{pmatrix}\begin{pmatrix}6\\8\end{pmatrix}$$

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4. Give an example of a map $f : \mathbb{R} \to \mathbb{R}$ which is $\boxed{\textit{not}}$ linear. What property does it not satisfy?

$f(x) = x^2 \qquad x \longmapsto x^2$

5. Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(x) = -5x$. Show that T is a linear transformation.

$\boxed{a,b>0}\ f(a+b) \overset{?}{=} f(a) + f(b)$

6. Consider the matrix
$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \qquad (a+b)^2 \overset{?}{=} a^2 + b^2$$
Define $T(x) = Ax$, and find the image of $u = \left(\begin{smallmatrix}5\\2\end{smallmatrix}\right)$, $v = \left(\begin{smallmatrix}1\\6\end{smallmatrix}\right)$, and $u + v$.

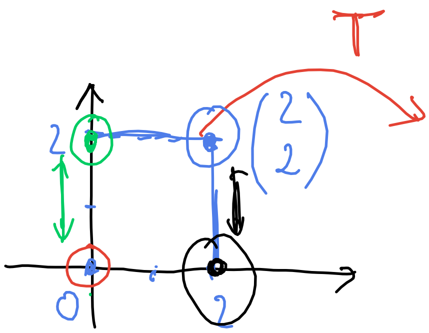
$\cancel{a^2} + 2ab + \cancel{b^2} \overset{?}{=} \cancel{a^2} + \cancel{b^2}$

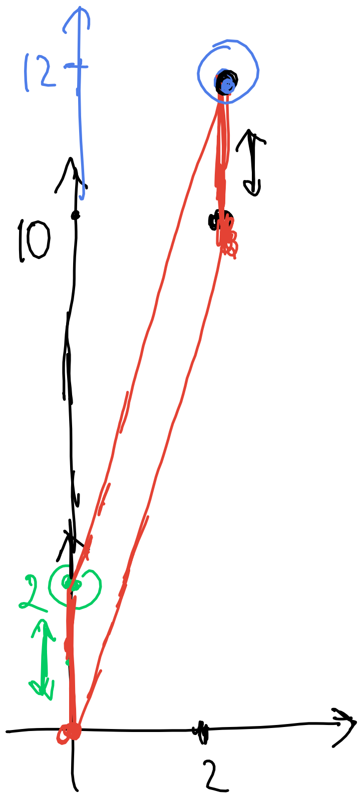
$2ab \overset{?}{=} 0$

3/ $A = \begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix}$

$Tx = Ax$

$(0,0)$ to $(2,2)$





$$\begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 10 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 12 \end{pmatrix}$$

turns a square
into a parallelogram.

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$\in \mathbb{R}^2$ $\in \mathbb{R}^2$

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ii) RHS: $cT(\vec{u}) = c\,T\begin{pmatrix} u_1 \\ v_2 \end{pmatrix} = c\left(-5\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right)$

$= c\begin{pmatrix} -5v_1 \\ -5v_2 \end{pmatrix} = \begin{pmatrix} -5cu_1 \\ -5cu_2 \end{pmatrix}$

~~$\$$~~LHS = RHS ✓

5/ *

$\boxed{i)}$ $T(u+v) = T(u) + T(v)$

for all $u, v \in \mathbb{R}^2$

ii) $T(c\vec{u}) = cT(\vec{u})$

scalar = numbers

for all $u \in \mathbb{R}^2$, c scalar.

$\vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathbb{R}^2$

$T(\vec{u} + \vec{v})$ $\quad \vec{u} + \vec{v} = \begin{pmatrix} u_1 + v_2 \\ v_1 + v_2 \end{pmatrix}$

$= T\begin{pmatrix} u_1 + u_2 \\ v_1 + v_2 \end{pmatrix}$

LHS.

$= -5\begin{pmatrix} u_1 + v_2 \\ v_1 + v_2 \end{pmatrix} = \begin{pmatrix} -5u_1 - 5v_2 \\ -5v_1 - 5v_2 \end{pmatrix}$

$T(u) = T\begin{pmatrix} u_1 \\ v_2 \end{pmatrix} = -5\begin{pmatrix} u_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -5v_1 \\ -5v_2 \end{pmatrix}$

+

$T(v) = T\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = -5\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -5v_1 \\ -5v_2 \end{pmatrix}$

$T(u) + T(v) = \begin{pmatrix} -5u_1 \\ -5v_2 \end{pmatrix} + \begin{pmatrix} -5v_1 \\ -5v_2 \end{pmatrix}$

$= \begin{pmatrix} -5v_1 - 5v_1 \\ -5v_2 - 5v_2 \end{pmatrix} \rightsquigarrow$ RHS

LHS = RHS ✓

ii) $c\vec{u} = c\begin{pmatrix} u_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} cu_1 \\ cv_2 \end{pmatrix}$

$T(c\vec{u}) = T\begin{pmatrix} cu_1 \\ cv_2 \end{pmatrix} = -5\begin{pmatrix} cu_1 \\ cv_2 \end{pmatrix}$

$= \begin{pmatrix} -5cu_1 \\ -5cv_2 \end{pmatrix}$

LHS