Math 31 - Fall 2021 - Discussion 8

1. As a warm-up, show that the columns e_1, e_2, e_3, e_4 of the 4×4 identity matrix span $\mathbb{R}^4$. This works in the same way for any n, not just for $n = 4$.

2. Let $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and let $\boxed{T : \mathbb{R}^2 \to \mathbb{R}^4}$ be a linear transformation such that
$$T(e_1) = \begin{pmatrix} 2 \\ -5 \\ 1 \\ 4 \end{pmatrix} \text{ and } T(e_2) = \begin{pmatrix} -1 \\ -1 \\ 7 \\ 3 \end{pmatrix}.$$
Find a formula for the image of an arbitrary vector x in $\mathbb{R}^2$.

3. Find a matrix corresponding to a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that first stretches vectors horizontally by a factor of 2 and then reflects them across the vertical axis. Hint: draw a picture first.

4. Find an example of a linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$ (for a particular choice of m and n) that
   (a) is both one-to-one and onto,
   (b) is one-to-one but not onto,
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5. Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be a linear transformation corresponding with the matrix
$$A = \begin{pmatrix} 1 & -4 & 8 & 1 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$
Does T map onto $\mathbb{R}^3$? Is it one-to-one?

1/ $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$\underbrace{\ }_{\mathbb{R}^4} \ \underbrace{\ }_{\mathbb{R}^4}$

$\{e_1, e_2, e_3, e_4\}$ spans $\mathbb{R}^4$

linear indep:

$x_1 = x_2 = x_3 = x_4 = 0$

2/ $T\underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{\vec{x} \in \mathbb{R}^2} = \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}$

$A = \begin{bmatrix} | & | \\ T(e_1) & T(e_2) \\ | & | \end{bmatrix}$

$A = \begin{bmatrix} 2 & -1 \\ -5 & -1 \\ 1 & 7 \\ 4 & 3 \end{bmatrix}$

$Tx = Ax$

$= \begin{bmatrix} 2 & -1 \\ -5 & -1 \\ 1 & 7 \\ 4 & 3 \end{bmatrix} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\vec{x} \in \mathbb{R}^2} = \underbrace{\begin{bmatrix} 2x_1 - x_2 \\ -5x_1 - x_2 \\ x_1 + 7x_2 \\ 4x_1 + 3x_2 \end{bmatrix}}$

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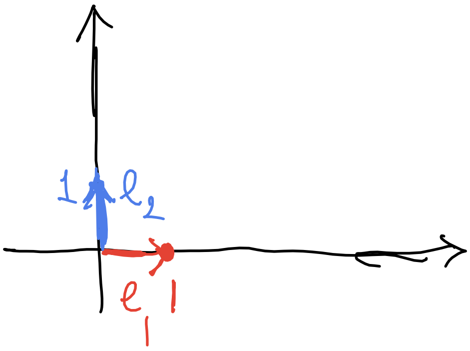

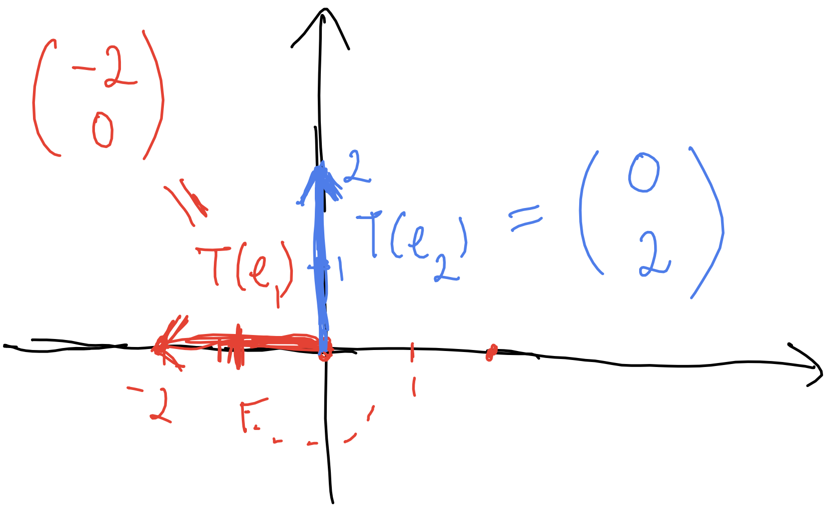

$T(e_1) = ?$ $\quad$ $T(e_2) = ?$

$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad$ $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$A = \begin{bmatrix} T(e_1) & T(e_2) \end{bmatrix}$$

$$A = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}$$

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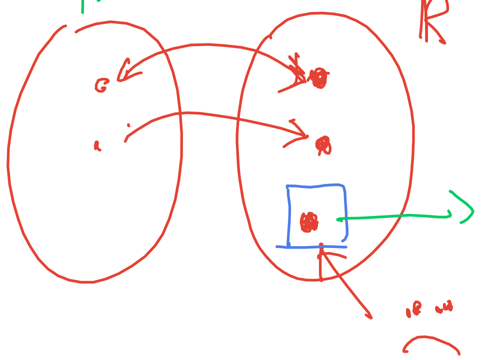
4/a/ $T: \mathbb{R}^2 \to \mathbb{R}^2$ $\quad \vec{x}$

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Joseph

$$\to \underbrace{\begin{bmatrix} x_2 \\ x_1 \end{bmatrix}}$$

b/



b/ → d: Shamari

$$\underset{A}{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}} \underset{x}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} = \begin{bmatrix} x_2 \\ 0 \end{bmatrix}$$

not onto: $\boxed{\text{not } 1\text{-}1}$

$\begin{pmatrix} 5 \\ 6 \end{pmatrix}$ in Range

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$$

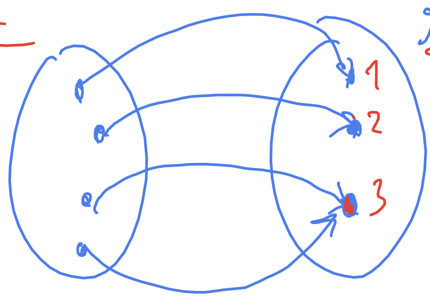
$$Ax = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$$

cannot find x

$\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ in Range

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

ONTO:



$$T: \mathbb{R}^2 \to \mathbb{R}^2 : A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$A \underbrace{\begin{bmatrix} 5 \\ 2 \end{bmatrix}}_{x_2} = \underset{Ax_2}{\begin{bmatrix} 2 \\ 0 \end{bmatrix}} \qquad A \underbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{x_1} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

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4/ b/ one to one, not onto:

$$T: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$

$$\begin{bmatrix} x \\ y \end{bmatrix} \longmapsto \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$$

- $\begin{bmatrix} 5 \\ 6 \\ 1 \end{bmatrix}$ has no $x \in \mathbb{R}^2$ st $Tx = \begin{bmatrix} 5 \\ 6 \\ 1 \end{bmatrix}$

$\Rightarrow$ NOT onto

- one-to-one:

$\begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$ then $\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}$

$$T\vec{x} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$$

this is 1 to 1.

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c/ onto, not 1 to 1:

$$T: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longrightarrow \begin{pmatrix} x \\ y \end{pmatrix}$$

onto: $\begin{pmatrix} x \\ y \end{pmatrix}$ then $\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}, \begin{pmatrix} x \\ y \\ 2 \end{pmatrix}$ output

in Range

$$T\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = T\begin{pmatrix} x \\ y \\ 2 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

not 1 to 1.

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$$A = \begin{pmatrix} \boxed{1} & 0 & 0 & 13 \\ 0 & \boxed{1} & -2 & 3 \\ 0 & 0 & 0 & \boxed{1} \end{pmatrix}$$
1st row ✓
✓
3rd row ✓

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$$A = \begin{pmatrix} 1 & -4 & 8 & 1 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix} : \mathbb{R}^4 \longrightarrow \underline{\underline{\mathbb{R}^3}}$$

if & only if

<u>Theorem 12, p. 78</u>: 1/ one to one $\Longleftrightarrow$ column of A <u>linear indept</u>

2/ onto $\Longleftrightarrow$ column of A span $\mathbb{R}^3$ (range)

$$A = \begin{pmatrix} 1 & \overset{0}{-4} & 8 & 1 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 13 & 0 \\ 0 & 1 & -2 & 3 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 \end{array}\right) \begin{array}{l} x_1 = 0 \\ \\ x_4 = 0 \end{array}$$

$x_2 - 2x_3 = 0 \Rightarrow x_2 = 2x_3$: free x_3

$x_3 = 5 \Rightarrow x_2 = 10$

<u>Not</u> all $x = 0 \Rightarrow$ <u>not</u> indep $\Rightarrow$ <u>not</u> 1 to 1.

2/ <u>onto</u>: Theorem 4, p. 37: column of A span $\mathbb{R}^3$

if A has pivot <u>position on every row</u>

column of A span $\mathbb{R}^3 \Rightarrow$ A is <u>onto</u>.