Applied Linear Algebra - MATH 31
UC Riverside - Fall 2021 - Practice Midterm Exam

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(1) Let A be the matrix below and b be the vector given below.

$$A = \begin{pmatrix} 1 & -4 & 8 & 1 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad b = \begin{pmatrix} 6 \\ 2 \\ 1 \end{pmatrix}$$

(1a) Write the reduced row echelon form of A.
(1b) Write the general solution of the system $Ax = b$.
(1c) Write the general solution of the system $Ax = 0$.

(2) Find the matrix representing the linear transformations described below.
(2a) $T : \mathbb{R}^2 \to \mathbb{R}^2$ is the rotation by an angle of 90 degrees counterclockwise.
(2b) $G : \mathbb{R}^2 \to \mathbb{R}^2$ is given by reflection on the x-axis.

(3) Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation given by $T(x) = Ax$ for all x in $\mathbb{R}^4$, where

$$A = \begin{pmatrix} 1 & -4 & 8 & 1 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(3a) Does T map $\mathbb{R}^4$ onto $\mathbb{R}^3$?
(3b) Is T one-to-one?

(4) Let

$$A = \begin{pmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{pmatrix}, \quad u = \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \quad b = \begin{pmatrix} 3 \\ 2 \\ -5 \end{pmatrix}, \quad c = \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix}.$$

(4a) Find $T(u)$.
(4b) Find all vectors x in $\mathbb{R}^2$ such that $T(x) = b$.
(4c) Is c in the range of T?

(5) Let

$$A = \begin{pmatrix} 1 & -3 \\ 3 & 5 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -3 \\ 3 & 5 \end{pmatrix}.$$

(5a) Find AB.
(5b) Find A^{-1}

1/ a/

$$\left(\begin{array}{cccc|c} 1 & \boxed{-4} & 8 & 1 & 6 \\ 0 & 1 & -2 & 3 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right) \longrightarrow \text{turn into } 0$$

$$\xrightarrow{4R_2 + R_1 \to R_1} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 13 & 14 \\ 0 & 1 & -2 & \boxed{3} & 2 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right) \longrightarrow \text{turn into } 0$$

$$\xrightarrow{-3R_3 + R_2 \to R_2} \left(\begin{array}{cccc|c} 1 & 0 & 0 & \boxed{13} & 14 \\ 0 & 1 & -2 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right) \longrightarrow \text{turn into } 0$$

$$\xrightarrow{-13R_3 + R_1 \to R_1} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & -2 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

A reduced row echelon $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

1/b/ $Ax = b$

$$\left(\begin{array}{cccc|c} \boxed{1 & 0 & 0 & 0 & 1} \\ 0 & \boxed{1} & -2 & 0 & -1 \\ 0 & 0 & 0 & \boxed{1} & 1 \end{array}\right)$$

$x_1 = 1$

$x_4 = 1$

pivot:

$x_2 - 2x_3 = -1$

$\Rightarrow x_2 = 2x_3 + 1$

with x_3 free:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2x_3 + 1 \\ x_3 \\ 1 \end{bmatrix}$$

1/c: $Ax = 0$

If we put augmented $[A \mid 0]$

note that whatever we do with the elementary operation on "0", it is still "0".

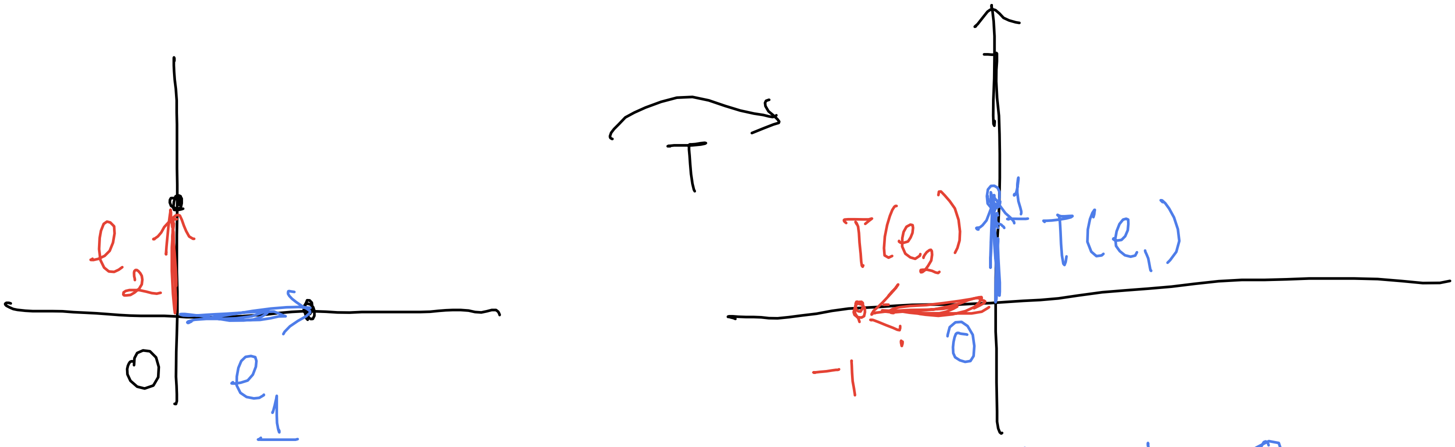
So: $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$ $\Rightarrow x_1 = 0$

$x_4 = 0$

$$1x_2 - 2x_3 = 0 \Rightarrow x_2 = 2x_3, \; x_3 \text{ free}$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 2x_3 \\ x_3 \\ 0 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

2/a/ $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$



rotated 90°
counter clockwise

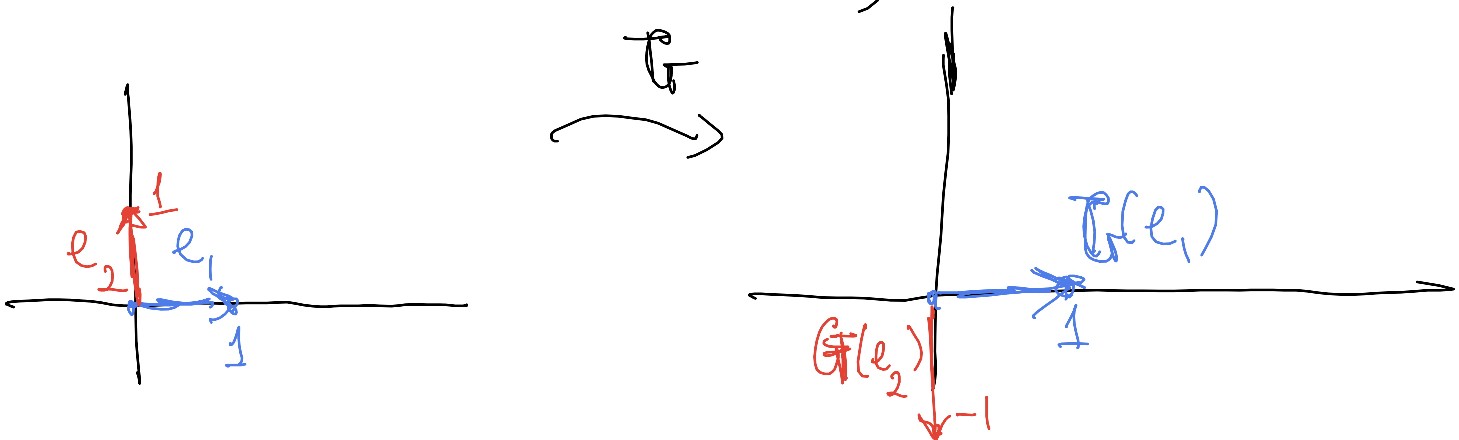
so $T(e_1) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$T(e_2) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

So matrix representing T is: $\begin{bmatrix} | & | \\ T(e_1) & T(e_2) \\ | & | \end{bmatrix}$

$= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

2/b/ $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$



- reflexsive across x-axis: $G(e_1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

- $G(e_2) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$
  reflexise over x-axis

So matrix representing G: $\begin{bmatrix} | & | \\ G(e_1) & G(e_2) \\ | & | \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

3/ $T: \mathbb{R}^4 \to \mathbb{R}^3$ by $T(x) = Ax$

$$A = \begin{pmatrix} 1 & -4 & 8 & 1 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

<u>Theorem 12</u>: T maps onto $\mathbb{R}^3$ if and only if column of A spans $\mathbb{R}^3$.

<u>Theorem 8</u>: column of A span $\mathbb{R}^3$ if A has <u>3</u> pivots. This A is same as in problem 1 we saw it has <u>3</u> pivots. So column of A spans $\mathbb{R}^3$. Hence T maps <u>onto</u> $\mathbb{R}^3$.

b/ Theorem 12: (part b)

T maps one-to-one if and only if column of A are linearly indep, in $\mathbb{R}^3$.

Theorem 8:

we have 4 columns, meaning 4 vectors, in $\mathbb{R}^3$, since $4 > 3$, any set with 4 elements in $\mathbb{R}^3$ is linearly dependent.

So column of A is lineary dependent

$\Rightarrow$ column of A not (linearly) independent

$\Rightarrow$ T is $\boxed{\text{NOT}}$ onto.

4/

a/ $Tu = Au$

$$= \begin{pmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} \boxed{8} \\ \boxed{1} \\ \boxed{-9} \end{pmatrix}$$

$1 \cdot 2 + (-3) \cdot (-1) = 8$

$3 \cdot 2 + 5 \cdot (-1) = 1$

$-1 \cdot 2 + 7 \cdot (-1) = -9$

b/ Find $\vec{x}$ st $T\vec{x} = b$, means solve $\vec{x}$ s.t. $A\vec{x} = b$: Augmented Matrix

turn into 0

$$\left(\begin{array}{cc|c} 1 & -3 & 3 \\ \boxed{3} & 5 & 2 \\ -1 & 7 & -5 \end{array}\right) \xrightarrow[\to R_2]{3R_2 + R_3} \left(\begin{array}{cc|c} 1 & -3 & 3 \\ 0 & 26 & -13 \\ -1 & 7 & -5 \end{array}\right)$$

divide R_2 by -13 to get small numbers:

$$\left(\begin{array}{cc|c} 1 & -3 & 3 \\ 0 & -2 & 1 \\ \boxed{-1} & 7 & -5 \end{array}\right) \xrightarrow{R_1 + R_3 \to R_3} \left(\begin{array}{cc|c} 1 & -3 & 3 \\ 0 & -2 & 1 \\ 0 & 4 & -2 \end{array}\right)$$

pivots

$$\xrightarrow[\to R_3]{2R_2 + R_3} \left(\begin{array}{cc|c} 1 & -3 & 3 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{array}\right)$$

$$-2x_2 = 1 \Rightarrow x_2 = \frac{-1}{2}$$

$$x_1 - 3x_3 = 3$$

$$x_1 = 3 + 3x_3 = 3 - \frac{3}{2} = \frac{3}{2}.$$

$$\vec{x} = \begin{pmatrix} \frac{3}{2} \\ -\frac{1}{2} \end{pmatrix}$$

4/c/ Is $c = \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix}$ in the range?

mean can we find $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ st $A\vec{x} = c$? Augmented Matrix:

$$\left(\begin{array}{cc|c} 1 & -3 & 3 \\ \boxed{3} & 5 & 2 \\ -1 & 7 & 5 \end{array}\right) \xrightarrow{3R_3 + R_2 \to R_2} \left(\begin{array}{cc|c} 1 & -3 & 3 \\ 0 & 26 & 10 \\ \boxed{-1} & 7 & 5 \end{array}\right)$$

$$\xrightarrow{R_1 + R_3 \to R_3} \left(\begin{array}{cc|c} 1 & -3 & 3 \\ 0 & \boxed{26} & \boxed{10} \\ 0 & \boxed{4} & \boxed{8} \end{array}\right) \Rightarrow 4x_2 = 8$$

$$\Rightarrow x_2 = 2$$

But $26x_2 = 10 \Rightarrow x_2 = \frac{10}{26} \neq 2$

Not possible (not consistent)

$\Rightarrow$ no $\vec{x}$ $\Rightarrow$ c $\boxed{\text{not}}$ in Range of $T_{\boxed{A}}$

5/a) $AB = \begin{pmatrix} 1 & -3 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} 1 & -3 \\ 3 & 5 \end{pmatrix} = \begin{pmatrix} -8 & -18 \\ 18 & 16 \end{pmatrix}$

$$1 \cdot 1 + -3 \cdot 3 = -8$$

$$3 \cdot 1 + 5 \cdot 3 = 18$$

$$1 \cdot (-3) + -3 \cdot 5 = -18$$

$$3 \cdot (-3) + 5 \cdot 5 = 16$$

b) $A^{-1} = \dfrac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

$$= \frac{1}{1 \cdot 5 - (-3)(3)} \begin{pmatrix} 5 & 3 \\ -3 & 1 \end{pmatrix} = \frac{1}{14} \begin{pmatrix} 5 & 3 \\ -3 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{5}{14} & \frac{3}{14} \\ \frac{-3}{14} & \frac{1}{14} \end{pmatrix}$$

$\square$