

Math 46 - Fall 2021 - Discussion section 002 and 005

Instructor: Prof. Feng Xu

Summary of some methods to solve ODE

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## 1 2.4. Transformation of Non Linear Equations into Separable Equations

**Exercise 1.** A Bernoulli differential equation is one of the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n.$$

Observe that, if  $n = 0$  or  $1$ , the Bernoulli equation is linear. For other values of  $n$ , the substitution  $u = y^{1-n}$  transforms the Bernoulli equation into the linear equation

$$\frac{du}{dx} + (1 - n)P(x)u = (1 - n)Q(x).$$

Use an appropriate substitution to solve the equation

$$xy' + y = 9xy^2,$$

and find the solution that satisfies  $y(1) = -5$ .



## 1 2.4. Transformation of Non Linear Equations into Separable Equations

**Exercise 1.** A Bernoulli differential equation is one of the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n \quad n=2$$

Observe that, if  $n = 0$  or  $1$ , the Bernoulli equation is linear. For other values of  $n$ , the substitution  $u = y^{1-n}$  transforms the Bernoulli equation into the linear equation

$$\frac{du}{dx} + (1-n)P(x)u = (1-n)Q(x).$$

Handwritten notes for the substitution:

$$u = y^{1-2} = y^{-1}$$

$$u' = -y^{-2} y' = -\frac{y'}{y^2}$$

Use an appropriate substitution to solve the equation

$$xy' + y = 9xy^2, \quad y' + \frac{1}{x}y = 9y^2$$

and find the solution that satisfies  $y(1) = -5$ .

(divide by  $y^2$ ):

$$\frac{y'}{y^2} + \frac{1}{x}y^{-1} = 9$$

$$u' + \frac{1}{x}u = 9$$

$$u' + \frac{1}{x}u = 0 \implies \frac{u'}{u} = -\frac{1}{x} \implies \text{integral}$$

$$\ln|u| = \ln|x|^{-1} \implies u = x^{-1}$$

$$u = v \cdot x^{-1}$$

$$u' = v' x^{-1} - v x^{-2}$$

$$+ \frac{1}{x} u = \frac{1}{x} v x^{-1} = v x^{-2}$$

$$q = v' x^{-1}$$

$$v' = q x$$

$$v = \frac{q x^2}{2} + c$$

$$u = v \cdot x^{-1} = \left( \frac{q x^2}{2} + c \right) x^{-1}$$

$$y^{-1} = \left( \frac{q x^2}{2} + c \right) x^{-1}$$

$$(-5)^{-1} = \left( \frac{q}{2} + c \right) \cdot 1 \Rightarrow c = -\frac{1}{25} - \frac{q}{2} = -4.54$$

$$y^{-1} = \left( \frac{q x^2}{2} - 4.54 \right) x^{-1}$$

$$y = \frac{1}{y^{-1}} = \frac{1}{\left( \frac{q x^2}{2} - 4.54 \right) x^{-1}}$$

$$y = \frac{x}{\frac{q x^2}{2} - 4.54}$$

**Exercise 2.** A Bernoulli differential equation is one of the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n \quad (*)$$

Observe that, if  $n = 0$  or  $1$ , the Bernoulli equation is linear. For other values of  $n$ , the substitution  $u = y^{1-n}$  transforms the Bernoulli equation into the linear equation

$$\frac{du}{dx} + (1 - n)P(x)u = (1 - n)Q(x).$$

Consider the initial value problem

$$xy' + y = -4xy^2, \quad y(1) = -5.$$

- (a) This differential equation can be written in the form (\*) with  $P(x) = ?$ ,  $Q(x) = ?$ , and  $n = ?$
- (b) The substitution  $u = ?$  will transform it into the linear equation  $\frac{du}{dx} + ?u = ?$
- (c) Using the substitution in part (b), we rewrite the initial condition in terms of  $x$  and  $u$ :  $u(1) = ?$
- (d) Now solve the linear equation in part (b), and find the solution that satisfies the initial condition in part (c).  $u(x) = ?$
- (e) Finally, solve for  $y$ .  
 $y(x) = \underline{\hspace{4cm}}$ .



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 $y(x) = \text{_____}$ .





**Exercise 3.** A Bernoulli differential equation is one of the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n.$$

Observe that, if  $n = 0$  or  $1$ , the Bernoulli equation is linear. For other values of  $n$ , the substitution  $u = y^{1-n}$  transforms the Bernoulli equation into the linear equation

$$\frac{du}{dx} + (1 - n)P(x)u = (1 - n)Q(x).$$

Use an appropriate substitution to solve the equation

$$y' - \frac{6}{x}y = \frac{y^4}{x^{16}},$$

and find the solution that satisfies  $y(1) = 1$ .



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## 2 Separable

**Exercise 1.** Use the "mixed partials" check to see if the following differential equation is exact.

If it is exact find a function  $F(x,y)$  whose differential,  $dF(x,y)$  is the left hand side of the differential equation. That is, level curves  $F(x,y) = C$  are solutions to the differential equation:

$$\underbrace{(-1x^4 + 2y)}_M dx + \underbrace{(2x + 3y^2)}_N dy = 0$$

First:

$$M_y(x,y) = ?, \text{ and } N_x(x,y) = ?$$

If the equation is not exact, enter *not exact*, otherwise enter in  $F(x,y)$  here \_\_\_\_\_

$$\begin{array}{l} M_y = 2 \\ N_x = 2 \end{array} \left. \vphantom{\begin{array}{l} M_y = 2 \\ N_x = 2 \end{array}} \right\} \text{equal : } \underline{\underline{\text{exact}}}$$

$$M_y = 2 : \int M dx = -\frac{x^5}{5} + 2xy + f(y) = F$$



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$$(2xy^2 + 3y)dx + (2x^2y + 3x)dy = 0$$

First:  $M_y(x,y) = \underline{\hspace{2cm}}$  , and  $N_x(x,y) = \underline{\hspace{2cm}}$ .

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If it is exact find a function  $F(x, y)$  whose differential,  $dF(x, y)$  gives the differential equation. That is, level curves  $F(x, y) = C$  are solutions to the differential equation:

$$\frac{dy}{dx} = \frac{x^3 - y}{x + 4y^4}$$

First rewrite as

$$M(x, y) dx + N(x, y) dy = 0$$

where  $M(x, y) = \underline{\hspace{2cm}}$  ,  
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**Exercise 4.** Use the "mixed partials" check to see if the following differential equation is exact.

If it is exact find a function  $F(x, y)$  whose differential,  $dF(x, y)$  is the left hand side of the differential equation.

That is, level curves  $F(x, y) = C$  are solutions to the differential equation

$$(-4e^x \sin(y) + 2y)dx + (2x + 4e^x \cos(y))dy = 0$$

First, if this equation has the form  $M(x, y)dx + N(x, y)dy = 0$ :

$M_y(x, y) = \underline{\hspace{2cm}}$  , and  $N_x(x, y) = \underline{\hspace{2cm}}$ .

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**Exercise 5.** The differential equation

$$y - 3y^2 = (y^6 + x) y'$$

can be written in differential form:

$$M(x, y) dx + N(x, y) dy = 0$$

where

$$M(x, y) = \text{_____}, \text{ and } N(x, y) = \text{_____}.$$

The term  $M(x, y) dx + N(x, y) dy$  becomes an exact differential if the left hand side above is divided by  $y^2$ . Integrating that new equation, the solution of the differential equation is \_\_\_\_\_ =  $C$ .



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