Math 46 - Fall 2021 - Discussion section 002 and 005
Instructor: Prof. Feng Xu
Summary of some methods to solve ODE

Prepared by: Khoi Vo

October 8, 2021

1 2.4. Transformation of Non Linear Equations into Separable Equations

Exercise 1. A Bernoulli differential equation is one of the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n.$$

Observe that, if $n = 0$ or 1, the Bernoulli equation is linear. For other values of n, the substitution $u = y^{1-n}$ transforms the Bernoulli equation into the linear equation

$$\frac{du}{dx} + (1-n)P(x)u = (1-n)Q(x).$$

Use an appropriate substitution to solve the equation

$$xy' + y = 9xy^2,$$

and find the solution that satisfies $y(1) = -5$.

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1 2.4. Transformation of Non Linear Equations into Separable Equations

Exercise 1. A Bernoulli differential equation is one of the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n.$$

$n = 2$

Observe that, if $n = 0$ or 1, the Bernoulli equation is linear. For other values of n, the substitution $u = y^{1-n}$ transforms the Bernoulli equation into the linear equation

$$\frac{du}{dx} + (1-n)P(x)u = (1-n)Q(x).$$

$u = y^{1-2} = y^{-1}$

$u' = -y^{-2}y' = -\frac{y'}{y^2}$

Use an appropriate substitution to solve the equation

$$xy' + y = 9xy^2,$$

$y' + \frac{1}{x}y = 9y^2$

and find the solution that satisfies $y(1) = -5$.

divide by y^2:

$\frac{y'}{y^2} + \frac{1}{x}y^{-1} = 9$

$u' + \frac{1}{x}u = 9$

$u' + \frac{1}{x}u = 0 \Rightarrow \frac{u'}{u} = -\frac{1}{x} \Rightarrow$ integral

$\ln|u| = \ln|x|^{-1}$

$\Rightarrow u = x^{-1}$

$$u = v \cdot x^{-1}$$

$$u' = v' x^{-1} - v x^{-2}$$

$$+\frac{1}{x} u = \frac{1}{x} v x^{-1} = v x^{-2}$$

$$9 = v' x^{-1}$$

$$v' = 9x$$

$$v = \frac{9x^2}{2} + C$$

$$\boxed{u = v \cdot x^{-1} = \left(\frac{9x^2}{2} + C\right) x^{-1}}$$

$$y^{-1} = \left(\frac{9x^2}{2} + C\right) x^{-1}$$

$$(-5)^{-1} = \left(\frac{9}{2} + C\right) \cdot 1 \Rightarrow C = -\frac{1}{25} - \frac{9}{2} = -4.54$$

$$y^{-1} = \left(\frac{9x^2}{2} - 4.54\right) x^{-1}$$

$$y = \frac{1}{y^{-1}} = \frac{1}{\left(\frac{9x^2}{2} - 4.54\right) x^{-1}}$$

$$y = \frac{x}{\frac{9x^2}{2} - 4.54}$$

Exercise 2. A Bernoulli differential equation is one of the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n \qquad (*)$$

Observe that, if $n = 0$ or 1, the Bernoulli equation is linear. For other values of n, the substitution $u = y^{1-n}$ transforms the Bernoulli equation into the linear equation

$$\frac{du}{dx} + (1-n)P(x)u = (1-n)Q(x).$$

Consider the initial value problem

$$xy' + y = -4xy^2, \quad y(1) = -5.$$

(a) This differential equation can be written in the form $(*)$ with $P(x) = ?$, $Q(x) = ?$, and $n = ?$

(b) The substitution $u = ?$ will transform it into the linear equation $\frac{du}{dx} + ?u = ?$

(c) Using the substitution in part (b), we rewrite the initial condition in terms of x and u: $u(1) = ?$

(d) Now solve the linear equation in part (b), and find the solution that satisfies the initial condition in part (c). $u(x) = ?$

(e) Finally, solve for y.

$y(x) =$ ________________________.

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(d) Now solve the linear equation in part (b), and find the solution that satisfies the initial condition in part (c). $u(x) =?$

(e) Finally, solve for y.

$y(x) =$ ________________________.

Exercise 3. A Bernoulli differential equation is one of the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n.$$

Observe that, if $n = 0$ or 1, the Bernoulli equation is linear. For other values of n, the substitution $u = y^{1-n}$ transforms the Bernoulli equation into the linear equation

$$\frac{du}{dx} + (1-n)P(x)u = (1-n)Q(x).$$

Use an appropriate substitution to solve the equation

$$y' - \frac{6}{x}y = \frac{y^4}{x^{16}},$$

and find the solution that satisfies $y(1) = 1$.

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2 Separable

Exercise 1. Use the "mixed partials" check to see if the following differential equation is exact.

If it is exact find a function F(x,y) whose differential, $dF(x,y)$ is the left hand side of the differential equation. That is, level curves $F(x,y) = C$ are solutions to the differential equation:

$$(\underbrace{-1x^4 + 2y}_{M})dx + (\underbrace{2x + 3y^2}_{N})dy = 0$$

First:

$M_y(x,y) =$?, and $N_x(x,y) =$?

If the equation is not exact, enter *not exact,* otherwise enter in $F(x,y)$ here __________

$M_y = 2$
$N_x = 2$
} equal : exact

2 Separable

Exercise 1. Use the "mixed partials" check to see if the following differential equation is exact.

If it is exact find a function F(x,y) whose differential, $dF(x,y)$ is the left hand side of the differential equation. That is, level curves $F(x,y) = C$ are solutions to the differential equation:

$$(-1x^4 + 2y)dx + (2x + 3y^2)dy = 0$$

First:

$M_y(x,y) =?$, and $N_x(x,y) =?$

If the equation is not exact, enter *not exact,* otherwise enter in $F(x,y)$ here ____________

Exercise 2. Use the "mixed partials" check to see if the following differential equation is exact.
If it is exact find a function F(x,y) whose differential, $dF(x, y)$ is the left hand side of the differential equation. That is, level curves $F(x, y) = C$ are solutions to the differential equation:

$$(2xy^2 + 3y)dx + (2x^2y + 3x)dy = 0$$

First: $M_y(x, y) =$ ____________ , and $N_x(x, y) =$ ____________.
If the equation is not exact, enter *not exact,* otherwise enter in $F(x, y)$ here ____________

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Exercise 3. Use the "mixed partials" check to see if the following differential equation is exact.

If it is exact find a function $F(x, y)$ whose differential, $dF(x, y)$ gives the differential equation. That is, level curves $F(x, y) = C$ are solutions to the differential equation:

$$\frac{dy}{dx} = \frac{x^3 - y}{x + 4y^4}$$

First rewrite as

$$M(x, y)\, dx + N(x, y)\, dy = 0$$

where $M(x, y) =$ ______________ ,

and $N(x, y) =$ ______________.

If the equation is not exact, enter *not exact,* otherwise enter in $F(x, y)$ as the solution of the differential equation here ____________________ $= C$.

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If the equation is not exact, enter *not exact,* otherwise enter in $F(x, y)$ as the solution of the differential equation here ________________________ $= C$.

Exercise 4. Use the "mixed partials" check to see if the following differential equation is exact.

If it is exact find a function $F(x, y)$ whose differential, $dF(x, y)$ is the left hand side of the differential equation.

That is, level curves $F(x, y) = C$ are solutions to the differential equation

$$(-4e^x \sin(y) + 2y)dx + (2x + 4e^x \cos(y))dy = 0$$

First, if this equation has the form $M(x, y)dx + N(x, y)dy = 0$:

$M_y(x, y) =$ ____________ , and $N_x(x, y) =$ ____________.

If the equation is not exact, enter *not exact,* otherwise enter in $F(x, y)$ here ____________

Exercise 4. Use the "mixed partials" check to see if the following differential equation is exact.

If it is exact find a function $F(x, y)$ whose differential, $dF(x, y)$ is the left hand side of the differential equation.

That is, level curves $F(x, y) = C$ are solutions to the differential equation

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If the equation is not exact, enter *not exact,* otherwise enter in $F(x, y)$ here ____________

Exercise 5. The differential equation

$$y - 3y^2 = \left(y^6 + x\right) y'$$

can be written in differential form:

$$M(x, y)\, dx + N(x, y)\, dy = 0$$

where
$M(x, y) =$ _______________ , and $N(x, y) =$ _______________.

The term $M(x, y)\, dx + N(x, y)\, dy$ becomes an exact differential if the left hand side above is divided by y^2. Integrating that new equation, the solution of the differential equation is ______________________ $= C$.

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