Math 245, Fall 2021

Analytic Number Theory

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A proof of the prime number theory using the Tchebychev ψ function

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Abstract

In this paper, we summarize the work of a proof of the Prime Number Theorem using the Chebychev function ψ. The ψ function is defined as the sum of all the $\log p$ such that $p^m \le x$. We will show that $\psi(x) \sim x$ as $x \to \infty$, will imply $\pi(x) \sim x/\log(x)$ as $x \to \infty$, in which the second statement is the content of the Prime Number Theorem. In fact, we can obtain this from the asymptotic of a cousin function of the function ψ which we will define in the content of this paper.

1 preliminary definitions

Definition 1.1 (Tchebychev's function)**.** We define the Tchebychev's function as followed

$$\psi(x) := \sum_{p^m \le x} \log p$$

We define another function based on this, says $\Lambda(n)$, which will equal ti $\log p$ if $n = p^m$ for some prime p and positive $m > 1$, and $\Lambda(n) = 0$ otherwise. Then it is clear that

$$\psi(x) = \sum_{1 \le n \le x} \Lambda(n)$$

For convenience, we also note that if $p^m \le x$, then $m \le \log(x)/\log(p)$, thus we have another formula for our Tchebychev function:

$$\psi(x) = \sum_{p \le x} \left[\frac{\log x}{\log p} \right] \log p$$

where for any real number a, the notation $[a]$ means the largest integer that is less than or equal to a.

2 the preparation for the main results

Some of the theorems of estimating $\zeta(s)$ will be used for the main results, so in this session we will prepare the readers with those theorems.

Theorem 2.1. There exists a sequence of entire functions $\delta_n(s)$ for $n \geq 1$ such that $|\delta_n(s)| \leq \frac{|s|}{n^{\sigma+1}}$ (again, $s = \sigma + it$), and that

$$\sum_{1 \leq n \leq N} \frac{1}{n^s} - \int_1^N \frac{ds}{x^s} = \sum_{1 \leq n \leq N} \delta_n(s)$$

for any integer $N > 1$.

Proof. We compare the series $\sum_{1 \leq n \leq N} \frac{1}{n^s}$ with $\sum_{1 \leq n \leq N} \int_n^{n+1} \frac{1}{x^s} dx$ and denote:

$$\delta_n(s) := \int_n^{n+1} \left(\frac{1}{n^s} - \frac{1}{x^s} \right) dx$$

The Mean value Theorem in Complex Form: for $f : \mathbb{R} \to \mathbb{C}$ dfifferentiable complex function, then for all $a, b \in \mathbb{R}$ we get:

$$\left| \frac{f(b) - f(a)}{b - a} \right| \leq |f'(c)| \text{ for some } c \in (a, b).$$

Applied mean value theorem in Complex form to $f(x) = x^{-s}$, for any $n \leq x$ (and also $x \leq n + 1$):

$$\left| \frac{1}{n^s} - \frac{1}{x^s} \right| \leq \max_{n \leq y \leq x} \frac{d}{dx} (|x^{-s}|)|_{x=y}$$

Then note that

$$\frac{d}{dx}(x^{-s}) = \frac{-s}{x^{s+1}}$$

Taking the absolute value gives:

$$\frac{d}{dx}(|x^{-s}|) \leq \frac{|s|}{|x^{s+1}|} \leq \frac{|s|}{|n^{s+1}|} = \frac{|s|}{n^{\sigma+1}}$$

In short we showed the integrand of $\delta_n(s)$ has a bound:

$$\left| \frac{1}{n^s} - \frac{1}{x^s} \right| \leq \frac{|s|}{|n^{s+1}|} = \frac{|s|}{n^{\sigma+1}}$$

And therefore

$$|\delta_n(s)| \leq \int_n^{n+1} \left| \frac{1}{n^s} - \frac{1}{x^s} \right| dx \leq \frac{|s|}{n^{\sigma+1}}$$

And note that:

$$\int_1^N \frac{dx}{x^s} = \sum_{1 \leq n < N} \int_n^{n+1} x^{-s} dx$$

so

$$\begin{aligned}\sum_{1\le n<N}\frac{1}{n^s}-\int_1^N\frac{dx}{x^s} &= \sum_{1\le n<N}\frac{1}{n^s}-\sum_{1\le n<N}\int_n^{n+1}x^{-s}dx\\ &= \sum_{1\le n<N}\int_n^{n+1}\frac{1}{n^s}dx-\sum_{1\le n<N}\int_n^{n+1}x^{-s}dx\\ &= \sum_{1\le n<N}\int_n^{n+1}\left(\frac{1}{n^s}-\frac{1}{x^s}\right)\\ &= \sum_{1\le n<N}\delta_n(s)\end{aligned}$$

■

Theorem 2.2. For $\Re(s)=\sigma>0$ we have:

$$\zeta(s)-\frac{1}{s-1}=H(s)$$

where $H(s)$ is a holomorphic function in the half-plane $Re(s)>0$.

Proof. By previous theorem 2.1, we let N going to infinity in the formula:

$$\sum_{1\le n\le N}\frac{1}{n^s}-\int_1^N\frac{ds}{x^s}=\sum_{1\le n\le N}\delta_n(s)$$

And note that by the estimate $|\delta_n(s)|\le\frac{|s|}{n^{\sigma+1}}$, we have the uniform convergence of the series $\sum\delta_n(s)$. Since $\sigma>1$, the $\sum n^{-s}$ converges, and this proved the theorem for $\sigma>1$. By the uniform convergence of $\sum_{\delta}{}_n(s)$ which is holomorphic when $\Re(s)>0$, we can extend ζ to this half-plane and that the formula continues to hold. ■

Theorem 2.3. Suppose $s=\sigma+it$ with $\sigma,t\in\mathbb{R}$. Then for each σ_0 with $0\le\sigma_0\le1$ and for every $\epsilon>0$, there exists a constant c_ϵ so that:
a. $|\zeta(s)|\le c_\epsilon|t|^{1-\sigma_0+\epsilon}$ and $|t|\ge1$.
b. $|\zeta'(s)|\le c_\epsilon|t|^{\epsilon}$ and $|t|\ge1$.

Proof. For the first part, from previous theorem we know that $|\delta_n(s)|\le|s|/n^{\sigma+1}$. In addition, when $x\ge n$ we have another estimate for $\delta_n(s)$

as followed:

$$\begin{aligned}
|\delta_n(s)| &\leq \int_n^{n+1} \left| \frac{1}{n^s} - \frac{1}{x^s} \right| dx \\
&\leq \int_n^{n+1} \frac{1}{|n^s|} + \frac{1}{|x^s|} dx \\
&\leq \int_n^{n+1} \frac{1}{|n^s|} + \frac{1}{|n^s|} dx \\
&= \int_n^{n+1} \frac{1}{n^\sigma} + \frac{1}{n^\sigma} dx = \frac{2}{n^\sigma}
\end{aligned}$$

Then:

$$\begin{aligned}
|\delta_n(s)| &= |\delta_n(s)|^\delta |\delta_n(s)|^{1-\delta} \\
&\leq \left(\frac{|s|}{n^{\sigma_0+1}} \right)^\delta \left(\frac{2}{n^{\sigma_0}} \right)^{1-\delta} \\
&\leq \frac{2|s|^\delta}{n^{\sigma_0+\delta}}
\end{aligned}$$

Now choose $\delta = 1 - \sigma_0 + \epsilon$ amd use the absolute value to the equation

$$\zeta(s) - \frac{1}{s-1} = \sum_n \delta_n(s)$$

to get:

$$|\zeta(s)| \leq \left| \frac{1}{s-1} \right| + 2|s|^{1-\sigma_0+\epsilon} \sum_{n \geq 1} \frac{1}{n^{1+\epsilon}}$$

This is the content of the estimate of ζ.
For the second part, by Cauchy integral:

$$\zeta'(s) = \frac{1}{2\pi r} \int_0^{2\pi} \zeta(s + re^{i\theta}) e^{i\theta} d\theta$$

Use the first part estimate and integration is on the circle center at s radius $r = \epsilon$ will give the result. ■

3 the main result

The Prime Number Theorem concerns about the asymptotic of the numbers of primes less than or equal to x, namely the function $\pi(x)$.

Theorem 3.1 (Prime Number Theorem)**.**

$$\pi(x) \sim \frac{x}{\log x}$$

as $x \to \infty$.

Here is the main result of this paper that relating the asymptotic of $\psi(x)$ to the asymptotic of $\pi(x)$, which in turns will give us the Prime Number Theorem:

Theorem 3.2. If $\psi(x) \sim x$ as $x \to \infty$ then we will have $\pi(x) \to x/\log(x)$ as $x \to \infty$.

Proof. We first have the following:

$$\psi(x) = \sum_{p \leq x} \left[\frac{\log x}{\log p}\right] \log p \leq \sum_{p \leq x} \frac{\log x}{\log p} \log p = \pi(x) \log x$$

Divide by x gives:

$$\frac{\psi(x)}{x} \leq \frac{\pi(x) \log x}{x}$$

Since $\psi(x) \sim x$, taking the infimum gives:

$$1 \leq \liminf_{x \to \infty} \pi(x) \frac{\log x}{x}$$

Now fix an $0 < \alpha < 1$, note:

$$\psi(x) \geq \sum_{p \leq x} \log p \geq (\pi(x) - \pi(x^{\alpha})) \log x^{\alpha}$$

The above is last inequality is because we are counting the number of prime less than x and bigger than x^{α}, also note that $\log p \geq \log x^{\alpha}$. So we get:

$$\psi(x) + \alpha\pi(x^{\alpha}) \log x \geq \alpha\pi(x) \log x$$

Divide by x we get:

$$\frac{\psi(x)}{x} + \alpha\pi(x^{\alpha}) \frac{\log x}{x} \geq \alpha \frac{\pi(x) \log x}{x}$$

Note that $\pi(x^{\alpha}) \leq x^{\alpha}$ and $\alpha < 1$ and $\psi(x) \sim x$, then:

$$1 \geq \alpha \limsup_{x \to \infty} \pi(x) \frac{\log x}{x}$$

Taking $\alpha \to 1$ gives:

$$1 \geq \limsup_{x\to\infty} \pi(x)\frac{\log x}{x}$$

So because $1 \geq \limsup \geq \liminf \geq 1$, hence we get the limit exists and

$$\lim_{x\to\infty} \pi(x)\frac{\log x}{x} = 1$$

■

We will now attempt to show that $\psi(x)$ is asymptotic to x as $x \to \infty$, but not to anyone suprise (but me), we will use a cousin function of ψ to get our result. Here is the function that we are going to use:

Definition 3.3 (The cousin of Chebychev function)**.**

$$\psi_1(x) := \int_1^x \psi(y)dy$$

And here is one of the key theorem about the asymptotic of this cousin function which will give us the asymptotic of the original Chebychev function:

Theorem 3.4. If $\psi_1(x) \sim \frac{x^2}{2}$ as $x \to \infty$, then $\psi(x) \sim x$.

Proof. Take $\alpha < 1 < \beta$. Since ψ is an increasing function, note that:

$$\psi(x) \leq \frac{1}{(\beta-1)x}[\psi_1(\beta x) - \psi_1(x)]$$

This is obtained using the Mean value theorem for function $\psi_1(x)$ on $[x, \beta x]$.

Then dividing by x gives:

$$\frac{\psi(x)}{x} \leq \frac{1}{(\beta-1)x}\left[\frac{\psi_1(\beta x)}{(\beta x)^2}\beta^2 - \frac{\psi_1(x)}{x^2}\right]$$

Taking the $\limsup$ give:

$$\limsup_{x\to\infty} \frac{\psi(x)}{x} \leq \frac{1}{(\beta-1)x}\left[\frac{1}{2}\beta^2 - \frac{1}{2}\right] = \frac{1}{2}(\beta+1)$$

Since this is true for all $\beta > 1$, it must be true that $\limsup_{x\to\infty} \frac{\psi(x)}{x} \leq 1$. A completely similar argument with α gives $\liminf_{x\to\infty} \frac{\psi(x)}{x} \geq 1$. Hence the limit exists and:

$$\lim_{x\to\infty} \frac{\psi(x)}{x} = 1$$

■

Remark 3.5. Theorem 3.4 will give us asymptotic of $\psi(x)$ from the asymptotic of $\psi_1(x)$, which then using theorem 3.2 will give us the asymptotic of $\pi(x)$, which is the Prime Number Theorem.

It is not by chance that in earlier section we introduce some estimate of the $\zeta(s)$ function, in this next subsection, we would like to relate $\psi_1(x)$ (and hence ψ) to the $\zeta(s)$ function:

3.1 relationship of $\zeta(s)$ and $\psi_1(s)$

We first note the following useful formula which will be stated as a lemma:

Lemma 3.6. Let $s - \sigma + it$ with $\sigma > 1$ then:

$$\frac{\zeta'(s)}{\zeta(s)} = -\sum_{n=1}^{\infty} \Lambda(n) n^{-s} = -\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}$$

Proof. From the Euler product formula we get:

$$\zeta(s) = \prod_p \frac{1}{1 - p^{-s}}$$

Since on $\sigma > 1$ which is contained in the principal domain of the $\log(z)$ complex logarith function, we can take the log and still get an analytic function on this, hence we get:

$$\log \zeta(s) = -\sum_p \log(1 - p^{-s})$$

Also note that the above equation is possible on the principal branch of the log function (ie, we are allowed to use $\log(ab) = \log(a) + \log(b)$ and also from the convergence of the infinite product, the log product formula can be extended to infinite product). As the function on the left is analytic on the half plane $\sigma > 1$, we can take derivatives of both sides and get:

$$\begin{aligned}
\frac{\zeta'(s)}{\zeta(s)} &= \frac{d}{ds}\left(-\sum_p \log(1-p^{-s})\right) \\
&= -\sum_p \left(\frac{d}{ds}\log(1-p^{-s})\right)
\end{aligned}$$

interchange of sum and derivatives is ok due to analyticity of log

$$\begin{aligned}
&= -\sum_p \frac{\frac{d}{ds}(1-p^{-s})}{1-p^{-s}} \\
&= -\sum_p \frac{-\frac{d}{ds}(p^{-s})}{1-p^{-s}} \\
&= \sum_p \frac{\frac{d}{ds}(p^{-s})}{1-p^{-s}} \\
&= \sum_p \frac{-\log(p)p^{-s}}{1-p^{-s}} \\
&= -\sum_p \log(p)\left(\frac{1}{1-p^{-s}} - 1\right) \\
&= -\sum_p \log(p)\sum_{n=1}^{\infty}(p^{-s})^n, \text{ using geometric series with } p^{-s} \\
&= -\sum_p \log(p)\sum_{n=1}^{\infty} p^{-ns} \\
&= -\sum_p \log(p)\sum_{n=1}^{\infty}(p^n)^{-s}
\end{aligned}$$

In short we get:

$$\frac{\zeta'(s)}{\zeta(s)} = -\sum_p \log(p)\sum_{n=1}^{\infty}(p^n)^{-s}$$

Just to be a little bit cleaner in the formula above, recall the Von Mangold function

$$\Lambda(n) := \log p$$

if there is a p prime such that $p^k = n$ and 0 otherwise. Then with $n = p^k$ satisfying Λ function, we get:

$$\sum_{n=1}^{\infty} \Lambda(n) n^{-s} = \sum_{p} (\log p) \sum_{n \geq 1} (p^n)^{-s}$$

and therefore we got:

$$\frac{\zeta'(s)}{\zeta(s)} = -\sum_{n=1}^{\infty} \Lambda(n) n^{-s}$$

■

Here is the main connection between $\psi_1(x)$ and $\zeta(s)$:

Theorem 3.7 (relationship between ζ and ψ)**.** For all $c > 1$ then:

$$\psi_1(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{x^{s+1}}{s(s+1)} \left(-\frac{\zeta'(s)}{\zeta(s)} \right) ds$$

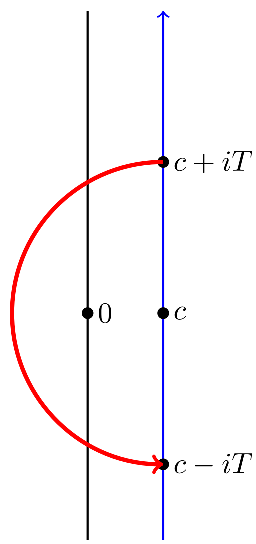
The proof of this theorem is quite lengthy, so we will break it into 1 main parts which is the part where we use a contour in this lemma:

Lemma 3.8. Again, let $c > 0$ then the following integral

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{a^s}{s(s+1)} ds$$

will either be zero if $0 < a < 1$, or has the value $1 - 1/a$ if $a \geq 1$.

Proof. We will show this lemma for the case $a \geq 1$ (which is our main use later) with the contour $\gamma(T)$ that consists of the vertical segment $S(T)$ from $c - iT$ to $c + iT$ and the half circle $C(T)$ centered at c of radius T lying to the left of the line. Here is the picture of the contour:



Denote the function

$$f(s) = \frac{a^s}{s(s+1)}ds$$

Then the residue of f at $s = 0$ is 1 and the residue of f at $s = -1$ is $-1/a$. Thus by Residue Theorem:

$$\frac{1}{2\pi i}\int_{\gamma(T)} f(s)ds = 1 - 1/a$$

Note that

$$\frac{1}{2\pi i}\int_{\gamma(T)} f(s)ds = \frac{1}{2\pi i}\int_{C(T)} f(s)ds + \frac{1}{2\pi i}\int_{\sigma=c} f(s)ds$$

So if we can show

$$\int_{C(T)} f(s)ds + \frac{1}{2\pi i} \to 0$$

when $T \to \infty$, then we are done. Note that $a^s = e^{\beta s}$ for $\beta = \log a$, then on $\sigma \le c$, the term $|a^s| \le e^{\beta c}$ while the denumerator of $f(s)$ which is $|s(s+1)| \ge (1/2)T^2$, so

$$|\int_{C(T)} f(s)ds| \le \frac{C}{T^2} 2\pi T \to 0$$

as $T \to \infty$. ∎

Now after we got from the lemma above, we will proceed with the proof of our theorem 3.7 above:

Proof. Recall the result of the lemma when $a \geq 1$:

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{a^s}{s(s+1)}ds = 1 - \frac{1}{a}$$

Using this lemma with $a = x/n \geq 1$, we obtain:

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{x^s}{n^s(s(s+1))}ds = 1 - \frac{n}{x}$$

Multiply both sides of the above by $x\sum_{n=1}^{\infty}\Lambda(n)$ (note that $\sum_{n=1}^{\infty}\Lambda(n)$ is finite) we get:

$$x\sum_{n=1}^{\infty}\Lambda(n)\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{x^s}{n^s(s(s+1))}ds = x\sum_{n=1}^{\infty}\Lambda(n)\left(1 - \frac{n}{x}\right)$$

For the left hand side we note:

$$\begin{aligned}
\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{x^s}{n^s(s(s+1))} &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x\sum_{n=1}^{\infty}\Lambda(n)\frac{x^s}{n^s(s(s+1))} \\
&= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \left(\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s}\right)\left(\frac{x^{s+1}}{n^s(s(s+1))}\right) \\
&= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \left(-\frac{\zeta'(s)}{\zeta(s)}\right)\left(\frac{x^{s+1}}{n^s(s(s+1))}\right)
\end{aligned}$$

In short, the left hand side is equal to:

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \left(-\frac{\zeta'(s)}{\zeta(s)}\right)\left(\frac{x^{s+1}}{n^s(s(s+1))}\right)$$

We have an observation:

$$\begin{aligned}\psi_1(x) &= \int_0^x \psi(y)dy \\ &= \int_0^x \sum_{n=1}^{\infty} \Lambda(n)\mathbf{1}_{n\leq y}(y)dy \\ &= \sum_{n\leq x} \int_0^x \Lambda(n)\mathbf{1}_{n\leq y}(y)dy \\ &= \sum_{n\leq x} \Lambda(n) \int_0^x \mathbf{1}(y)dy \\ &= \sum_{n\leq x} \Lambda(n)(x-n) \\ &= x\sum_{n=1}^{\infty} \Lambda(n)\left(1-\frac{n}{x}\right)\end{aligned}$$

So the right hand side of the equation in the beginning is $\psi_1(x)$. Thus we got:

$$\psi_1(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{x^{s+1}}{s(s+1)}\left(-\frac{\zeta'(s)}{\zeta(s)}\right)ds$$

And this completed the proof the theorem. ■

4 proof of the main result

As indicated in earlier section, the main result will be attained once we can find the asymptoptic of the cousin Chebychev function, that is to shown the following:

$$\psi_1(x) \sim \frac{x^2}{2}$$

as $x \to \infty$. We will dedicate this section to prove that result.
The idea is to use the theorem 3.7 above, which we will restate here for convenience:

Theorem 4.1. For all $c > 1$ then:

$$\psi_1(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{x^{s+1}}{s(s+1)}\left(-\frac{\zeta'(s)}{\zeta(s)}\right)ds$$

Let us denote

$$F(s) = \frac{x^{s+1}}{s(s+1)}\left(-\frac{\zeta'(s)}{\zeta(s)}\right)$$

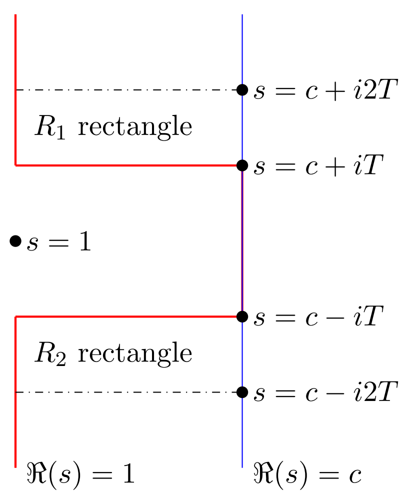
the integrand of the integral in the theorem for shorter notation. Our first claim would be as followed:

Proposition 4.2.

$$\psi_1(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)ds = \frac{1}{2\pi i}\int_{\gamma(T)} F(s)ds$$

where the contour $\gamma(T)$ has 1 segment on the line $\Re(s) = c$ from $-T \leq t \leq T$ and on the line $\Re(s) = 1$ consist of $T \leq t \leq \infty$ and $-\infty \leq t \leq -T$) (see picture below for clarification)

The red line in the picture below is the contour $\gamma(T)$.



Proof. Note that $\gamma(T)$ and the line $\Re(s) = 1$ already has the common section from $s = c + it$ for $-T \leq t \leq T$, so our goal is to show that the integral on the two (line arrays red) on the left is equal to the integration on the 2 line arrays blue.

We fix $c > 1$ and assume x is also fixed. We let T be chosen large (with the larger choice later specified). Let R_1 be the rectangle on top (see picture) and R_2 be rectangle on the bottom (see picture). On these two rectangles, $F(s)$ is analytic and so there integral is zero. Take R_1 as an example. Consider the direction as counterclockwise,

ie, on the blue line going up, then turn left, then going down on the vertical red line and going to the right on the horizontal red line. Let us denote:
a. The black dashed line going from the left $\alpha_1 := s = x + i2T$, where $1 \leq -x \leq c$.
b. The vertical red line going downward: $\alpha_2 := s = 1 + i2t$, where $T \leq -t \leq 2T$
c. The horizontal red line going to the right: $\alpha_3 := s = x + iT$, where $1 \leq x \leq c$.
d. The vertical blue line going upward: $\alpha_3 := s = c + it$, where $T \leq t \leq 2T$.
We have:

$$\int_{R_1} F(s)ds = 0 = -\int_{\alpha_1} F(s)ds + -\int_{\alpha_2} F(s)ds + \int_{\alpha_3} F(s)ds + \int_{\alpha_4} F(s)ds$$

So

$$\int_{\alpha_2} F(s)ds = \int_{\alpha_4} F(s)ds + \int_{\alpha_1} F(s)ds - \int_{\alpha_3} F(s)ds$$

The $\int_{\alpha_2} F(s)ds$ is integration on the left top red array and the $\int_{\alpha_4} F(s)ds$ is the integration on the top right blue array when letting $T \to \infty$. So if we can show that $\int_{\alpha_1} F(s)ds - \int_{\alpha_3} F(s)ds$ is small and goes to zero when we letting T goes to infinity, then we are done. Indeed, by the estimate of zeta function and its derivative in 2.3 We obtained:

$$\left|\frac{\zeta'(s)}{\zeta(s)}\right| \leq A|t|^{\eta}$$

for any fixed $\eta > 0$ whenever $\sigma \geq 1$ and $|t| \geq 1$. Thus

$$|F(s)| \leq B|t|^{-2+\eta}$$

in the rectangle R_1 (and also R_2) So:

$$\begin{aligned}\int_{\alpha_1} |F(s)|ds &\leq -\int_1^c B|t|^{-2+\eta}dx \\ &= B|2T|^{-2+\eta}(c-1) \to 0 \text{ as } T \to \infty\end{aligned}$$

and

$$\begin{aligned}\int_{\alpha_3} |F(s)|ds &\leq -\int_1^c B|t|^{-2+\eta}dx \\ &= B|T|^{-2+\eta}(c-1) \to 0 \text{ as } T \to \infty\end{aligned}$$

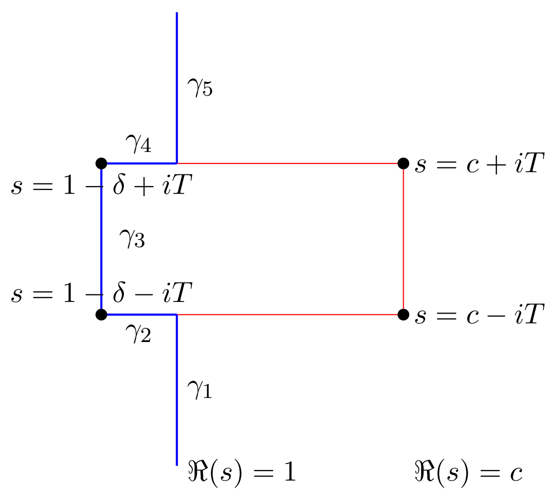
The bottom 2 arrays is shown similar using the bottom rectangle R_2 and this will completed this proposition. ■

Then we deform the contour $\gamma(T)$ to the contour $\gamma(T,\delta)$ (see picture for the contour $\gamma(T,\delta)$).

Proposition 4.3.

$$\frac{1}{2\pi i}\int_{\gamma(T)} F(s)ds = \frac{1}{2\pi i}\int_{\gamma(T,\delta)} F(s)ds + \frac{x^2}{2}$$

The following blue line is the contour $\gamma(T,\delta)$



Proof. For fixed T we choose small $\delta > 0$ so that ζ has no zero in the box

$$\{s = \sigma + it : 1-\delta \le \sigma \le 1, |t| \le T\}$$

Such choice of δ is possible since $\zeta(s)$ does not vanish on the line $\sigma = 1$. Then note that $F(s)$ has a simple pole at $s = 1$.By the estimate 2.2 we have:

$$\zeta(s) = \frac{1}{s-1} + H(s)$$

and so $\zeta'(s)/\zeta(s) = 1/(s-1) + h(s)$ for some $h(s)$ is holomorrphic near $s = 1$, so the residue of $F(s)$ at $s = 1$ is $\lim_{s\to 1}(s-1)F(s) = \frac{x^2}{2}$. Consequently:

$$\psi_1(x) = \frac{1}{2\pi i}\int_{\gamma(T)} F(s)ds = \frac{1}{2\pi i}\int_{\gamma(T,\delta)} F(s)ds + \frac{x^2}{2}$$

Or equivalently:

$$\psi_1(x) - \frac{x^2}{2} = \frac{1}{2\pi i}\int_{\gamma(T,\delta)} F(s)ds$$

We will now estimate$\int_{\gamma(T,\delta)} F(s)ds$ on each γ_i with $i = 1, 2, 3, 4, 5$ as denoted in the picture:
For $s \in \gamma_1$ (and hence similarly on γ_5) we have $\sigma = 1$ so $|x^{1+s}| = x^{1+\sigma} = x^2$. Then by the estimate of $\zeta'(s)$ and $\zeta(s)$ theorem 2.3 we have $|\zeta'(s)/\zeta(s)| \leq A|t|^{1/2}$. Thus :

$$|\int_{\gamma_1} F(s)ds| \leq Cx^2 \int_T^\infty \frac{|t|^{1/2}}{t^2} dt = Cx^2 \int_T^\infty \frac{1}{t^{3/2}}$$

Since the integral converges, there exitst T large so that the left hand side is $\leq \epsilon x^2/2$. A similar argument for γ_5.
For γ_3, we choose δ small and observe that $|x^{1+s}| = x^{1+1-\delta} = x^{2-\delta}$ hence there is a constant C_T (only depends on T) so that

$$|\int_{\gamma_3} F(s)ds| \leq C_T x^{2-\delta}$$

Lastly, for $s \in \gamma_2$ (and similarly on γ_4), the integral:

$$|\int_{\gamma_2} F(s)ds| \leq B_T \int_{1-\delta}^1 x^{1+\sigma} d\sigma \leq B_T \frac{x^2}{\log x}$$

In short, there are C_T, B_T such that:

$$|\psi_1(x) - \frac{x^2}{2}| = |\int_{\gamma(T,\delta)} F(s)ds| \leq \epsilon x^2 + C_T x^{2-\delta} + B_T \frac{x^2}{\log x}$$

Dividing by $x^2/2$ give:

$$|\frac{2\psi_1(x)}{x^2} - 1| \leq 2\epsilon + 2C_T x^{-\delta} + 2B_T \frac{1}{\log x}$$

Letting $x \to \infty$, $T \to \infty$ and $\delta \to 0$ will give the result. ■

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