Math 46 - Fall 2021 - Discussion section 002 and 005
Instructor: Prof. Feng Xu
Summary of some methods to solve ODE

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1 Disclaimer

This document is prepared in helping my students with their class. It certainly contain errors, and if you spot any error or have any comment about it, please do not hesititate to contact me at my email **khoi.vo@email.ucr.edu**.

Happy Learning together!!

Khoi.

2 Linear First Order Equations

If we have a form $y' + p(x)y = 0$ where $p(x)$ is a function of x only, this is called the **homogeneous** (linear first order equation).

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Example 1. Here is an example of a homogeneous (linear first order) ODE:

$$y' + \frac{1}{x}y = 0$$

In this example, $p(x) = \frac{1}{x}$.

If we did not get a 0 on the right hand side, but instead get a function of x, for instance, $f(x)$, and the ODE looks something like $y' + p(x)y = f(x)$, this is called **non-homogeneous**.

Example 2. Here is an example of a non-homogeneous (linear first order) ODE:

$$y' + \frac{1}{x}y = x^2$$

2.1 Homogeneous Linear First order

We will now summarize steps to solve for a homogeneous linear first order ODE. Let's use an example and try to solve that example:

Example 3. 1. Find a general solution y to this ODE:

$$y' + \frac{1}{x^2}y = 0$$

In this example, $p(x) = \frac{1}{x^2}$.

2. With the initial condition $y(1) = 1$, find the particular solution.

Solution. The textbook might do something such as the integral factor $I(x) = \int p(x)dx$. And in this case we can get the integration factor is $I(x) = \int \frac{1}{x} = \ln|x| + D$. Then the general solution will be $y = ce^{-I}$, which is

$$y = ce^{-I} = ce^{-(\ln|x|+D)} = ce^{x^{-1}} = ce^{-D}e^{-\ln|x|} = Ke^{-1}.$$

in which K is the constant (could be positive or negative) ce^{-D}

But I do not recommend doing this way, unless you can memorize the integration factor formula, and the general solution will be a constant multiply with e to the power of negative of that integration factor.

Step 1: Here is a way I usually do it with less memorization: **"Separate" x on one side and all the other y', y on the other side** to get:

$$\frac{y'}{y} = -\frac{1}{x}.$$

Step 2: Then you do integration, on the left hand side (everything is in y) so you do with respect to y, while on the left integrate with respect to x:

$$\ln(|y|) = -\ln|x| + C$$

Step 3: Now this has some technicality $|y|$ instead of y, but because we are raising exponent e of both sides, says:

$$|y| = e^{-\ln|x|+C} = e^C e^{\ln(|x|^{-1})} = e^C |x|^{-1}$$

We can think of e^C as some constant which could be positive or negative and write it as a K so that we can break the absolute values of y and x and get:

$$y = Kx^{-1}$$

And note that this is exactly the same as when we do the integration factor I.

Step 4: Solve for the constant K by the initial condition, that is, when $x = 1$, $y = 1$:

$$1 = K(1) = K$$

And we get the answer as

$$y = x^{-1}$$

2.2 Non-homogeneous Linear First Order

Let's use the earlier example:

Example 4. 1. Find a general solution y to this ODE:

$$y' + \frac{1}{x}y = x^2$$

2. With the initial condition $y(1) = 1$, find the particular solution.

Solution. I would like to introduce you with this idea below:

Step 1: This is an important idea in ODE, which is, to **start with the homogeneous case** and then build your actual solution of the non-homogeneous case based on the answer from the homogeneous case. So from the example in the previous part we get $y_1 = x^{-1}$ is a solution to the homogeneous case. Note that we do **NOT** include the constant K, as we will build our new solution with an unknown factor called $u(x)$ which might also have constants and variable x in it. We do not want to make the situation more complicated.

Step 2: Set

$$y = u(x)y_1 = u(x)x^{-1}$$

as a "guessed" solution to the non-homogeneous equation. Our job now will be find this $u(x)$. In words, we "scaled" the homogeneous solutions y_1 by a "factor" $u(x)$.

Step 3 : To find $u(x)$, we need to plug y back into the non-homogeneous equation. And first we need to find y'. Here, remember to use the **product rule** because y is the product of two functions of x the $u(x)$ and the x^{-1}.

$$y' = u'x^{-1} - ux^{-2}$$

Then we plug it into the non-homogeneous function Left Hand Side:

$$y' + \frac{1}{x}y = u'x^{-1} - ux^{-2} + \frac{1}{x}ux^{-1} = u'x^{-1}$$

since $\frac{1}{x}ux^{-1} = ux^{-2}$ so the last 2 terms canceled each other. Then compare this with the RHS (right hand side) of the non-homogeneous equation and we get:

$$u'x^{-1} = x^2$$

So

$$u'(x) = x^{-3}$$

Then we integration with respect to x and we can find $u(x) = \frac{x^{-2}}{-2} + C = \frac{1}{-2x^2} + C$

Step 4: Find the constant C:

$$y(x) = u(x)x^{-1} = (\frac{1}{-2x^2} + C)x^{-1}$$

When $x = 1$ then $y = 1$ gives:

$$1 = (\frac{1}{-2} + C)(1)$$

So $C = \frac{3}{2}$. Hence our particular solution to the non-homogeneous equation is

$$y = (\frac{1}{-2x^2} + \frac{3}{2})x^{-1} = \frac{1}{-2x^3} + \frac{3}{2x}$$

3 Separable

As the name said it all, you can usually separate the x and dx on one side and the other side will have only the y and dy. Then you just integrate each side correspondingly (wrt x on the side of dx and wrt y on the side of dy.

Example 5. Solve the equation $y' = x(1 + y^2)$

Solution. Rewrite the original equation a little bit into

$$\frac{y'}{1 + y^2} = x$$

Or

$$\frac{1dy}{1 + y^2} = xdx$$

Then integrate both sides:

$$\arctan(y) = \frac{x^2}{2} + C$$

And taking the tan give:

$$y = \tan(\frac{x^2}{2} + C)$$

4 Exact

Here I will use the example similar to the Midterm.

Example 6. Solve the equation:

$$(4x^3y^3 + 3x^2)dx + (3x^4y^2 + 6y^2)dy = 0$$

Solution. We will break this into some steps as followed:

Step 1 Identify the M and N term, say the M term is the term goes with the dx and the N term is the term that goes with the dy. So $M = 4x^3y^3 + 3x^2$ and $N = 3x^3y^2 + 6y^2$.

Step 2 This is an important step, that is, to check for exactness. Otherwise, you cannot solve the equation using exactness method. To check this you must check derivative of M with respect to y (**Pay attention, it is** y) and compare it with derivative of N with respect to x. If they are equal, then you have exact. If they are **not** equal, then you cannot use exact method. Here:

$$M_y = \frac{d}{dy}(4x^3y^3 + 3x^2) = 4x^3 \times 3y^2 = 12x^3y^2$$

Keep in mind derivatives w.r.t. y so x is considered as a constant.
On the other hand:

$$N_x = \frac{d}{dx}(3x^4y^2 + 6y^2) = 12x^3y^2$$

And so $M_y = N_x$ and this equation is exact. So you can proceed to next step.

Step 4 To get big F, we need to integrate Mdx with respect to x, so:

$$F(x, y) = \int Mdx = \int (4x^3y^3 + 3x^2)dx = x^4 \times y^3 + x^3 + g(y)$$

Very important: note the existence of $g(y)$ which is a function of y only and has no x. It is treated as the constant term in the integration of Mdx with respect to x. Constant in term of x can have y in it. Make sure you understand this idea. Now we will need to find this $g(y)$. To find this $g(y)$, we will differentiate the $F(x, y)$ we just found above, namely

$$F(x, y) = x^4y^3 + x^3 + g(y)$$

with respect to y and then compare it to N to find $g'(y)$ and hence help up find $g(y)$:

$$\frac{d}{dy}F(x, y) = 3y^2 \times x^4 + 0 + g'(y)$$

While the N term is $3x^4y^2 + 6y^2$, comparing this gives:

$$g'(y) = 6y^2$$

And so $g(y) = \int g'(y)dy = 2y^3 + K$ for some constant K (no x and y in it). Thus:

$$F(x, y) = x^4y^3 + x^3 + g(y) = x^4y^3 + x^3 + 2y^3 + K$$

5 second order linear

5.1 homogeneous

Given an equation of the type:

$$a_0y'' + a_1y' + a_2y = 0 \tag{1}$$

This is a second order linear homogeneous type. It is homogeneous because the right hand side is **zero**. And it is second order because we have y'' - second derivative on the left hand side.Steps to solve this type of equation are as followed:

Step 1 Set up the characteristic function:

$$a_0r^2 + a_1r + a_2 = 0$$

This is a quadratic equation with coefficients a_0, a_1, a_2 comes exactly from the given equation. Please keep in mind the order of these coefficients. The coefficient that goes with y'' is the same with the coeffiecient that goes with r^2. Since this is a quadratic, you can solve this and there are 3 cases possible solution of r:

Case 1: 2 distinct solutions, namely r_1 and r_2. In this case the given equation (1) will have the solutions e^{r_1x} and e^{r_2x} and we write it as:

$$y = c_1e^{r_1x} + c_2e^{r_2x}$$

You may ask if we can find c_1 and c_2, this is only possible when they give us some initial conditions like $y(0) = 1$ and $y'(0) = 2$, then we can plug in with $x = 0$ to get 2 equations and solve for c_1 and c_2.

Case 2 You only find 1 solution r_0, we say this solution r_0 has multiplicity 2. It is because the characteristic equation can be rewritten as $(r-r_0)^2 = 0$. The given equation (1) will now have $x \times e^{r_0x}$ as its solution. Be careful and note the x is being multiplied with e^{r_0x}. So

$$y = c_1xe^{r_0x}$$

is the solution. Again, finding c_1 is only possible when they give us some condition like $y(0) = 2$.

Case 3 You get 2 imaginary roots, say $r_1 = a + ib$ and $r_2 = a - ib$. Then your answer will be:

$$y = c_1e^{ax}\cos(bx) + c_2e^{ax}\sin(bx)$$

Example 7. Solve

$$y'' - ty' + 10y = 0$$

with initial condition $y(0) = -1$ and $y'(0) = 1$.

Solution. The characteristic equation is;

$$r^2 - 7r + 10 = 0$$

which could be factor into $(r-2)(r-5)$ and gives 2 solutions $r=2$ and $r=5$. Therefore the general solution is

$$y = c_1 e^{2x} + c_2 e^{5x}$$

Now to solve for c_1, c_2, we need to use $y(0) = -1$ and $y'(0) = 1$. When $x = 0$, $y = -1$ so plug into equation above gives:

$$-1 = c_1 + c_2$$

Also, to use $y'(0) = 1$ we need to take derivatives:

$$y'(x) = 2c_1 e^{2x} + 5c_2 e^{5x}$$

Hence plugging $x = 0$ and $y = 1$ gives

$$1 = 2c_1 + 5c_2$$

In short we have a system of equation:

$$\begin{cases} c_1 + \ \ c_2 = -1 \\ 2c_1 + 5c_2 = 1 \end{cases} \tag{2}$$

You can solve this system and get $c_1 = -2$ and $c_2 = 1$. And so the solution is

$$y = -2e^{2x} + e^{5x}$$

Example 8. Solve

$$y'' - 2y' + 2y = 0$$

with $y(0) = 3$ and $y'(0) = -2$.

Solution. The characteristic equation is

$$r^2 - 2r + 2 = 0$$

Which using quadratic formula has 2 imaginary roots:

$$r_1 = 1 + 1i, r_2 = 1 - 1i$$

Hence the general solution to this equation is (with $a = b = 14$)

$$y = c_1 e^{1x} \cos(1x) + c_2 e^{1x} \sin(1x)$$

To solve for c_1 and $c_2$plug in, and don't forget to take derivatives:

$$y'(x) = c_1 e^x \cos(x) + c_1 e^x(-\sin x) + c_2 e^x \sin x + c_2 e^x \cos x$$

When plugging $y'(0) = -2$ get

$$-2 = c_1 + c_2$$

Also when plug in $y(0) = 3$ gives:

$$3 = c_1$$

So we get $c_1 = 3$ and $c_2 = -5$ and the solution is

$$y = 3e^x \cos x - 5e^x \sin x$$

Example 9. Solve

$$y'' - 2y' + y = 0$$

with $y(0) = 7$ and $y'(0) = 4$.

Solution. The characteristic equation is

$$r^2 - 2r + 1 = 0$$

which can be factored into $(r-1)^2 = 0$ and get $r = 1$ as a root with multiplicative 2. Hence our general solution are e^{2x} and xe^{ex} (**do NOT forget the multiply with** x term):

$$y = c_1 e^{2x} + c_2 x e^{2x}$$

Again to find c_1 and c_2 we need a system of equation. First using $y(0) = 7$ we get:

$$7 = c_1$$

and taking derivative:

$$y'(x) = 2c_1 e^{2x} + c_2 e^{2x} + 2c_2 x e^{2x}$$

Then using $y'(0) = 4$ gives:

$$4 = 2c_1 + c_2$$

so $c_2 = 10$. Hence the solution is:

$$y = 7e^{2x} + 10xe^{2x}$$

5.2 non-homogeneous

The general equation looks like:

$$a_0 y'' + a_1 y' + a_2 y = f(x)$$

Step 1: Solve the homogeneous equation:

$$a_0 y'' + a_1 y' + a_2 y = 0$$

and get the solution to the homogeneous equation as in previous section and name it y_h.

Step 2: The most important (and the hardest) of equation of this type is to find the particular solution y_p. We will discuss that with examples later. In general, there are 2 ways of finding this y_p:

1. The method of undetermined coefficients
2. the method of variation of parameters

Step 3: The solution to the original equation would be

$$y = y_h + y_p$$

Example 10. Solve
$$y'' - 4y' + 5y = 1 + 5x$$

Solution. **Step 1:** Solve the homogeneous equation just like earlier section:

$$y'' - 4y' + 5y = 0$$

With characteristic function
$$r^2 - 4r + 5 = 0$$

This characteristic equation with quadratic formula and have imaginary solutions $r_1 = 2 + 1i$ and $r_2 = 2 - 1i$ and thus we got the solution (to the homogeneous part):

$$y_h = c_1 e^{2x} \cos x + c_2 e^{2x} \sin x$$

Step 2: Now onto finding the solution to the particular part. This is usually depends on the right hand side of the equation. Most of the time it is **guessing**. Here because the RHS is a polynomial of degree 1, we will guess the particular solution is also a polynomial of degree 1 and therefore it has the form:

$$y_p = A + Bx$$

With this guess, we can see that $y_p' = B$ and $y_p'' = 0$ so when we plug in the left hand side $y_p'' - 4y_p' + 5y$ we get:

$$0 - 4B + 5(A + Bx) = (5A - 4B) + 5Bx$$

Compare this with the right hand side $1 + 5x$, (ie, compare the coefficient of the power of x and the no power of x terms) we get

$$5A - 4B = 1$$

and
$$5B = 5$$

So $A = 1$ and $B = 1$. Thus $y_p = 1 + x$.

Step 3: The general solution to the original equation is:

$$y = y_h + y_p = c_1 e^{2x} \cos x + c_2 e^{2x} \sin x + 1 + x$$

Example 11. Solve
$$y'' - 7y' + 12y = 4e^{2x}$$

Solution. We start with the homogeneous equation:

$$y'' - 7y' + 12y = 0$$

And using characteristic equation as usual

$$r^2 - 7r + 12 = 0$$

This equation can be factored into $(r-3)(r-4)=0$ and so we get 2 roots $r=3$ and $r=4$. The solution to the homogeneous equation will look like:

$$y_h = c_1 e^{3x} + c_2 e^{4x}$$

Now onto the hard part, finding the particular solution. Here looking at the Right Hand Side, we **guess** the $y_p = Ae^{2x}$ (where the e^{2x} coming from the RHS). In this guess, we are lucky. See the next example, when we are not lucky and there will be some trouble later (and of course, solution to the trouble! Yay!). With this guess, we calculate: $y_p' = 2Ae^{2x}$ and $y_p'' = 4Ae^{2x}$. Then plug it into the left hand side:

$$y'' - 7y' + 12y = 4Ae^{2x} - 7(2Ae^{2x}) + 12(Ae^{2x}) = 2Ae^{2x}$$

And hence comparing it to the right hand side $4e^{2x}$ we get $2A = 4$ so $A = 2$. Thus

$$y_p = 2e^{2x}$$

Finally, our general solution will be:

$$y = y_h + y_p = c_1 e^{3x} + c_2 e^{4x} + 2e^{2x}$$

Example 12. Solve:

$$y'' - 7y' + 12y = 5e^{4x}$$

Solution. As usual, set up the homogeneous equation:

$$y'' - 7y' + 12y = 0$$

With the characteristic equation:

$$r^2 - 2r + 12 = 0$$

This characteristic has 2 solutions $r_1 = 3$ and $r_2 = 4$ and hence the homogeneous solution is:

$$y_h = c_1 e^{3x} + c_2 e^{4x}$$

Now we continue with the particular part. Looking at the right hand side $5e^{4x}$, we may want to guess $y_p = Ae^{4x}$.Unfortunately, because e^{4x} is the same with one of the solution of y_h, when we plug into the original equation trying to solve for A, we would not be able to find A. The fix/solution to problem **(You should know this!!)** is **multiply by power of x, starting with x. If it does not work, continue with x^2. Usually it should work**. So our guess would be

$$y_p = Axe^{4x}$$

With this guess, let's try calculating $y_p' = Ae^{4x} + 4Axe^{4x}$ and $y_p'' = 4Ae^{4x} + 4Ae^{4x} + 16Axe^{4x}$. Then we put them into the left hand side:

$$\begin{aligned} y'' - 7y' + 12y &= (4Ae^{4x} + 4Ae^{4x} + 16Axe^{4x}) - 7(Ae^{4x} + 4Axe^{4x}) + 12(Axe^{x}) \\ &= Ae^{4x} + 0Axe^{4x} \\ &= Ae^{4x} \end{aligned}$$

Comparing it with the RHS $5e^{4x}$ we get $A = 5$. And thus

$$y_p = 5xe^{5x}$$

So our general equation would be:

$$y = y_h + y_p = c_1e^{3x} + c_2e^{4x} + 5xe^{4x}$$

Example 13. Solve:

$$y'' - 8y' + 16y = 2e^{4x}$$

Solution. The homogeneous equation is:

$$y'' - 8y' + 16y = 0$$

And the characteristic equation is

$$r^2 - 8r + 16 = (r - 4)^2 = 0$$

So we get $r = 4$ with multiplicity 2, hence the solution to the homogeneous part is:

$$y_h = c_1e^{4x} + c_2xe^{4x}$$

As a reminder, please remember the xe^{4x} term.

Step 2: Find the y_p particular solution. Our guess again will base on the right hand side $2e^{4x}$ and start with $y_p = Ae^{4x}$, but that will not work as it is 1 of the solution of the homogeneous equation. We then try $y_p = xe^{4x}$ and unfortunately, it will **not** work again because it is also the solution of the homogeneous equation. So we must try $y_p = Ax^2e^{4x}$.[Note that we continue and multiply with x^2.] Then we will take derivatives $y_p' = 2Axe^{4x} + 4Ax^2e^{4x}$ and $y_p'' = 2Ae^{4x} + 8Axe^{4x} + 8Axe^{4x} + 16Ax^2e^{4x}$. We then put it back into the left hand side and get:

$$\begin{aligned} y'' - 8y' + 16y &= (2Ae^{4x} + 8Axe^{4x} + 8Axe^{4x} + 16Ax^2e^{4x}) - 8(2Axe^{4x} + 4Ax^2e^{4x}) + 16Ax^2e^{4x} \\ &= 0Ax^2e^{4x} + 0Axe^{4x} + 2Ae^{4x} = 2Ae^{4x} \end{aligned}$$

And comparing this to the RHS $2e^{4x}$ gives $A = 1$. Thus

$$y_p = x^2e^{4x}$$

So our final soution would be:

$$y = y_h + y_p = c_1e^{4x} + c_2xe^{4x} + x^2e^{4x}$$

Example 14. Solve:

$$y'' + 4y = 8\cos(2x) + 12\sin(2x)$$

Solution. Start with the homogeneous:

$$y'' + 4y = 0$$

with characteristic

$$r^2 + 4 = 0$$

This gives 2 imaginary roots $r_1 = 0 + 2i$ and $r_2 = 0 - 2i$, so the homogeneous solution is

$$y_h = c_1 e^{0x} \cos(2x) + c_2 e^{0x} \sin(2x) = c_1 \cos(2x) + c_2 \sin(2x)$$

Step 2: Find y_p. Since the right hand side has sum of $\cos(2x)$ and $\sin(2x)$ we guess that $y_p = A\cos(2x) + B\sin(2x)$. Unfortunately, this will **not** work because it is the same with our y_h, hence we must multiply by h and have this

$$y_p = x(A\cos(2x) + B\sin(2x))$$

And then find A and B. We have:

$$y_p' = A\cos(2x) + B\sin(2x) + x(-2A\sin(2x) + 2B\cos(2x))$$

and

$$y_p'' = -2A\sin(2x) + 2B\cos(2x) - 2A\sin(2x) + 2B\cos(2x) - 4Ax\cos(2x) - 4Bx\sin(2x)$$

In short:

$$y_p'' = -4A\sin(2x) + 4B\cos(2x) - 4Ax\cos(2x) - 4Bx\sin(2x)$$

Put into left hand side:

$$\begin{aligned} y'' + 4y &= (-4A\sin(2x) + 4B\cos(2x) - 4Ax\cos(2x) - 4Bx\sin(2x)) \\ &\quad + 4(Ax\cos(2x) + Bx\sin(2x)) \\ &= -4A\sin(2x) + 4B\cos(2x) \end{aligned}$$

comparing to the RHS $8\cos(2x) + 12\sin(2x)$ we get $-4A = 12$ and $4B = 8$, so $A = -3$ and $B = 2$. Then we got our particular solution:

$$y_p = x(-3\cos(2x) + 2\sin(2x))$$

And our final solution would be:

$$y = y_h + y_p = c_1\cos(2x) + c_2\sin(2x) + x(-3\cos(2x) + 2\sin(2x))$$

Lastly I would like to give an example of the variation of parameters method to find the particular solution:

Example 15. Solve $y'' - 2y' + 2y = 3e^x \sec(x)$

Solution. To start with the homogeneous part:

$$y'' - 2y' + 2 = 0$$

And the characteristic equation is

$$r^2 - 2r + 2$$

This has 2 imaginary roots $r_1 = 1 + i$ and $r_2 = 1 - i$, so the homogeneous solution is

$$y_h = c_1 e^x \cos(x) + c_2 e^x \sin(x)$$

Step 2: To find the particular solution y_p. This time we will use the method of variation parameter (**Note**: How do we know we should use this? When the right hand side is a complicated function, that is, **not** a polynomial like x^2+x+1, or not exponential function like e^{5x}, or not sum of sin and cos like $\cos(4x) + \sin(4x)$. Here we see $e^x \sec(x)$ which is very complicated.) We will build y_p based on y_h. Here we see that y_h has 2 parts the $e^x \cos(x)$ and the $e^x \sin(x)$. So the particular equation $y_p = u_1(e^x \cos(x)) + u_2(e^x \sin(x))$. And our goal will be finding u_1 and u_2. This method to find u_1 and u_2 depends on 2 equations as followed, and unfortunately, you have to memorize it:

$$u_1' y_1 + u_2' y_2 = 0$$

The first one is just u_1' and u_2' multiply with each solution of the homogeneous equation y_1 and y_2, so we get:

$$u_1' e^x \cos(x) + u_2' e^x \sin(x) = 0 \tag{3}$$

The second equation is a bit trickier to remember, it is also u_1' and u_2' but we multiply with the derivatives y_1' and y_2'. Here $y_1' = (e^x \cos x)' = e^x \cos x - e^x \sin x$ and $y_2' = (e^x \sin x)' = e^x \sin x + e^x \cos x$, and the right hand side is not zero but a fraction $\frac{f(x)}{p(x)}$ where $f(x)$ is the RHS of the original equation (here $f(x) = 3e^x \sec(x)$) and $p(x)$ is the coefficient of the y'' term which is 1 in this case, so we get the equation:

$$u_1'(e^x \cos x - e^x \sin x) + u_2'(e^x \sin x + e^x \cos x) = \frac{3e^x \sec(x)}{1} = 3e^x \sec(x) \tag{4}$$

In (3) we divide by e^x and get:

$$u_1' \cos(x) + u_2' \sin(x) = 0 \tag{5}$$

In (4) we also divide by e^x and get:

$$u_1'(\cos x - \sin x) + u_2'(\sin x + \cos x) = 3\sec(x) \tag{6}$$

In the above equation (6) we can take it and substract (5) and get:

$$-u_1' \sin(x) + u_2' \cos(x) = 3\sec x \tag{7}$$

Take a look at (5) and the new (7), in fact, take (5) multiply by $\sin x$ and added with (7) multiply by $\cos x$, the term $u_1' \sin x \cos x$ is cancelled and we are left with:

$$u_2'(\sin^2 x + \cos^2 x) = 3$$

So $u_2' = 3$ and thus $u_2 = 3x$. In addition, plug into (5) that $u_2' = 3$ then we get

$$u_1' = -u_2' \frac{\sin x}{\cos x} = -3 \tan x$$

So integrating $\tan x$ gives $u_1 = 3 \ln |\cos x|$.
And now we got our particular

$$y_p = u_1(e^x \cos x) + u_2(e^x \sin x) = 3 \ln |\cos x| e^x \cos x + 3xe^x \sin x$$

And for completeness, the general solution is

$$y = y_h + y_p = (c_1 e^x \cos(x) + c_2 e^x \sin(x)) + (3 \ln |\cos x| e^x \cos x + 3xe^x \sin x)$$

6 Higher order homogeneous linear differential equation

You know this type when you see you have like y''', or $y^{(4)}$ and so on. They are usually higher than 2-second order derivative of y. Also, you will see the right hand side with zero (as this is homogeneous). It is also important that we only solve for **homogeneous** type!

Example 16. Solve

$$y^{(3)} - 6y'' + 11y' - 6y = 0$$

Solution. Set up the characteristic equation:

$$r^3 - 6r^2 + 11r - 6 = 0$$

Solving this, making guess $r = 1$ is a solution, then you can (synthetic) divide to $r - 1$ and get $(r-2)(r-3)$ So we get 3 roots $r_1 = 1$, and $r_2 = 2$ and $r_3 = 3$, thus the answer is:

$$y = c_1 e^{1x} + c_2 e^{2x} + c_3 e^{3x}$$

Example 17. Solve $y''' + 3y'' + 3y' + y = 0$

Solution. You get a characteristic equation like:

$$r^3 + 3r^2 + 3r + 1 = 0$$

You guess $r = -1$ is a solution then divide it to $(r+1)$ and get $(r^2 + 2r + 1)$, or if you recognize this is $(r+1)^3$, you get 1 root $r = -1$ with multiplicity 3. Then you have 3 solutions: $y_1 = e^{-x}$ (when $r = -1$) and then you must multiply by x to get $y_2 = xe^{-x}$ and multiply by x^2 (because multiplicity of 3) to get $y_3 = x^2 e^{-x}$. That gives our general solution:

$$y = c_1 e^{-x} + c_2 x e^{-x} + c_3 x^2 e^{-2x}$$

Remark 18. whenever you have a multiplicity k, make sure you remember to multiply by x, x^2,..., until x^{k-1}. For example, if $k = 3$ like earlier example, you need up to x^2. If $k = 4$, then you need up to x^3. If $k = 2$, then you need up to x.

Example 19. Solve

$$y^{(4)} + 8y'' - 9y = 0$$

Solution. The characteristic equation is

$$r^4 + 8r^2 - 9$$

which you can solve by factoring and get:

$$(r^2 - 1)(r^2 + 9) = (r - 1)(r + 1)(r^2 + 9)$$

This gives you 4 roots $r_1 = 1$, $r_2 = -1$, $r_3 = 1 + 3i$, and $r_4 = 1 - 3i$ So the soutions are $y_1 = e^x$, $y_2 = e^{-x}$, $y_3 = e^x \cos 3x$ and $y_4 = e^x \sin 3x$. And the general soution is

$$y = c_1e^x + c_2e^{-x} + c_3e^x \cos 3x + c_4e^x \sin 3x$$

Example 20. Solve:

$$y^{(4)} + 2y'' + y = 0$$

Solution. The characteristic equation is

$$r^4 + 2r^2 + 1 = 0$$

which is $(r^2+1)^2 = 0$, so this gives $r_1 = 0+i$ of mulitilpicity 2 and $r_2 = -i$ of multiplicity 2. Here $a = 0$ and $b = 1$. Because of multiplicity 2, remember from remark to multiply by x, so the first two solutions are $y_1 = e^{0x} \cos(x) = \cos(x)$ and $y_2 = e^{0x} \sin x = \sin x$, then the other 2 solutions will be multiply by x of the first two solutions $y_3 = x \cos x$ and $y_4 = x \sin x$. Hence the general solution is

$$y = c_1 \cos x + c_2 \sin x + c_3x \cos x + c_4x \sin x$$