# 1 Disclaimer

This document is prepared in helping my students with their class. It certainly contain errors, and if you spot any error or have any comment about it, please do not hesititate to contact me at my email **khoi.vo@email.ucr.edu**.

Happy Learning together!!

Khoi.

# 2 Midterm

**Problem 1.** Solve

$$(4x^3y^3 + 3x^2)dx + (3x^4y^2 + 6y^2)dy = 0$$

*Solution.* We will break this into some steps as followed:

**Step 1** Identify the $M$ and $N$ term, say the $M$ term is the term goes with the $dx$ and the $N$ term is the term that goes with the $dy$. So $M = 4x^3y^3 + 3x^2$ and $N = 3x^3y^2 + 6y^2$.

**Step 2** This is an important step, that is, to check for exactness. Otherwise, you cannot solve the equation using exactness method. To check this you must check derivative of $M$ with respect to $y$ (**Pay attention, it is** $y$) and compare it with derivative of $N$ with respect to $x$. If they are equal, then you have exact. If they are **not** equal, then you cannot use exact method. Here:

$$M_y = \frac{d}{dy}(4x^3y^3 + 3x^2) = 4x^3 \times 3y^2 = 12x^3y^2$$

Keep in mind derivatives w.r.t. $y$ so $x$ is considered as a constant.
On the other hand:

$$N_x = \frac{d}{dx}(3x^4y^2 + 6y^2) = 12x^3y^2$$

And so $M_y = N_x$ and this equation is exact. So you can proceed to next step.

**Step 4** To get big $F$, we need to integrate $Mdx$ with respect to $x$, so:

$$F(x,y) = \int Mdx = \int (4x^3y^3 + 3x^2)dx = x^4 \times y^3 + x^3 + g(y)$$

**Very important:** note the existence of $g(y)$ which is a function of $y$ only and has no $x$. It is treated as the constant term in the integration of $Mdx$ with respect to $x$. Constant in term of $x$ can have $y$ in it. Make sure you understand this idea. Now we will need to find this $g(y)$. To find this $g(y)$, we will differentiate the $F(x,y)$ we just found above, namely

$$F(x,y) = x^4y^3 + x^3 + g(y)$$

with respect to $y$ and then compare it to $N$ to find $g'(y)$ and hence help up find $g(y)$:

$$\frac{d}{dy}F(x,y) = 3y^2 \times x^4 + 0 + g'(y)$$

While the $N$ term is $3x^4y^2 + 6y^2$, comparing this gives:

$$g'(y) = 6y^2$$

And so $g(y) = \int g'(y)dy = 2y^3 + K$ for some constant $K$ (no $x$ and $y$ in it). Thus:

$$F(x, y) = x^4y^3 + x^3 + g(y) = x^4y^3 + x^3 + 2y^3 + K$$

**Problem 2.** Solve

$$x^2y' = y^2 + xy - x^2$$

with $y(1) = 2$

*Solution.* If $x = 0$ then $y = 0$ is a solution. Consider $x \neq 0$. Then divide by $x^2$ gives:

$$y' = (\frac{y}{x})^2 + \frac{y}{x} - 1$$

Letting $y/x = u$, that means, $y = ux$, the above equation is:

$$(ux)' = u^2 + u - 1$$

Expanding the derivative on the LHS $(ux)' = u'x + u$ and putting it back:

$$u'x + u = u^2 + u - 1$$

So the $u$ term can be cancelled on both sides and we got:

$$u'x = u^2 - 1$$

This is separable of $u$ and $x$. Check to see that $u = \pm 1$ is a solution. Then consider when $u \neq \pm 1$ and devide by $u^2 - 1$ gives:

$$\frac{u'}{u^2 - 1} = x$$

or equivalently

$$\frac{du}{u^2 - 1} = xdx$$

Then we can integrate both sides:

$$\int \frac{du}{u^2 - 1} = \int xdx$$

The integration on the left hand side gives $\frac{x^2}{2} + c$, while the integration on the left needs a bit more work. Setting

$$\frac{1}{u^2 - 1} = \frac{A}{u - 1} + \frac{B}{u + 1}$$

and solve for $A$ and $B$. Multiply by $u^2 - 1$ gives:

$$1 = A(u + 1) + B(u - 1)$$

Letting $u = 1$ and $u = -1$ respectively in the equation give $A = \frac{1}{2}$ and $B = -\frac{1}{2}$. Thus

$$\frac{1}{u^2 - 1} = \frac{1}{2}\left(\frac{1}{u-1} - \frac{1}{u+1}\right)$$

Taking integration, and note that $\int \frac{1}{u-1} = \ln(u-1)$ and $\int \frac{1}{u+1} = \ln(u+1)$, so

$$\int \frac{1}{u-1} = \frac{1}{2}(\ln(u-1) - \ln(u+1) = \frac{1}{2}\ln(\frac{u-1}{u+1})$$

and this is the left hand side.
Putting LHS=RHS gives:

$$\frac{1}{2}\ln(\frac{u-1}{u+1}) = \frac{x^2}{2} + c$$

Multiply both sides by 2 and denote $M = 2c$ another constant:

$$\ln(\frac{u-1}{u+1}) = x^2 + M$$

Raising exponential $e$ power to both sides and write $K = e^M$ another constant gives:

$$\frac{u-1}{u+1} = e^{x^2}K$$

Then multiplying $u + 1$ to both sides and get:

$$u - 1 = (u+1)Ke^{x^2}$$

Expanding the RHS:

$$u - 1 = Ke^{x^2}u + Ke^{x^2}$$

So

$$u(1 - Ke^{x^2}) = 1 + Ke^{x^2}$$

and so

$$u = \frac{1 + Ke^{x^2}}{1 - Ke^{x^2}}$$

Note that $u = y/x$ so

$$\frac{y}{x} = \frac{1 + Ke^{x^2}}{1 - Ke^{x^2}}$$

Hence $y = \frac{x(1+Ke^{x^2})}{1-Ke^{x^2}}$ is a solution. Together with $y = 0$, $y = \pm x$ (since $u = \pm 1$) in the beginning. These are all solutions to this problem.

**Problem 3.** Solve

$$y' = x(1 - y^2)$$

*Solution.* Check that $y = 1$ and $y = -1$ are solutions to the problem. Then assume $y \neq \pm 1$. Then

$$\frac{y'}{1 - y^2} = x$$

So

$$\frac{dy}{1 - y^2} = xdx$$

Then integrate on both sides:

$$\int \frac{1}{1-y^2}dy = \int xdx$$

The integration on the right hand side is $\frac{x^2}{2} + c$.
Let's work on the integration on the left a bit careful. Recall from Calculus the method of breaking up

$$\frac{1}{1-y^2} = \frac{A}{1-y} + \frac{B}{1+y}$$

You can multiply by $1 - y^2$ and get

$$1 = A(1+y) + B(1-y)$$

Letting $y = 1$ and $y = -1$ respectively you can get $A = \frac{1}{2}$ and $B = \frac{1}{2}$. So

$$\frac{1}{1-y^2} = \frac{1}{2}\left(\frac{1}{1-y} + \frac{1}{1+y}\right)$$

Taking integration and note that $\int \frac{1}{1-y}dy = -\ln(1-y)$ and $\int \frac{1}{1+y}dy = \ln(1+y)$:

$$\int \frac{1}{1-y^2}dy = \frac{1}{2}\left(-\ln(1-y) + \ln(1+y)\right) = \frac{1}{2}\ln(\frac{1+y}{1-y})$$

Thus by putting LHS=RHS we got:

$$\frac{1}{2}\ln(\frac{1+y}{1-y}) = \frac{x^2}{2} + c$$

Multiplying both sides by 2 and replace $m = 2c$ another constant gives:

$$\ln(\frac{1+y}{1-y}) = x^2 + m$$

Raising exponential powers and write $e^m = K$ a constant give:

$$\frac{1+y}{1-y} = e^{x^2}K$$

Then

$$1 + y = (1-y)Ke^{x^2}$$

So

$$1 + y = Ke^{x^2} - yKe^{x^2}$$

So

$$1 - Ke^{x^2} = -y(1 + Ke^{x^2})$$

So

$$y = \frac{Ke^{x^2} - 1}{1 + Ke^{x^2}}$$

This is the answer, together with $y = \pm 1$ in the beginning.

**Problem 4.** Solve

$$y' + 2y = x^3 e^{-2x}$$

*Solution.* If you remember a formula, you can solve

$$y_1 = e^{-\int 2dx} = e^{-2x}$$

If you do not remember the formula, you can start with the homogeneous:

$$y' + 2y = 0$$

Then

$$\frac{dy}{dx} + 2y = 0$$

so

$$\frac{dy}{y} = -2dx$$

Then integration:

$$\log(y) = -2x$$

So $y_1 = e^{-2x}$, same as the formula.

Then we set

$$y = uy_1 = ue^{-2x}$$

And try to find $u$. We can start by doing derivatives $y' = (ue^{-2x})' = u'e^{-2x} - 2ue^{-2x}$ and so putting into the original equation, the left hand side is:

$$y' + 2y = (u'e^{-2x} - 2ue^{-2x}) + 2(ue^{-2x}) = u'e^{-2x}$$

Comparing with the right hand side:

$$u'e^{-2x} = x^3e^{-2x}$$

So $u' = x^3$ and therefore by integrating

$$u = \int x^3 dx = \frac{x^4}{4} + c$$

Hence:

$$y = ue^{-2x} = (\frac{x^4}{4} + c)e^{-2x}$$