3. $x\,\partial_x u + y\,\partial_y u = \boxed{2u}$

$u(x,y) = 1$ on unit circle $\{x^2+y^2=1\}$

Step 1: $a = x$, $\underbrace{b = y}$, $c = 2u$

Step 2: parametrize (initial)

$x_0(s) = \cos(s)$, $y_0(s) = \sin(s)$, $u_0(s) = 1$ ✓

Step 3:

$$\begin{cases} \partial_t x = \overset{a}{x}, & x_0(s) = \cos(s) \quad (1) \\ \partial_t y = y, & y_0(s) = \sin(s) \quad (2) \\ \partial_t u = 2u, & u_0(s) = 1 \quad (3) \end{cases}$$

(1) $\partial_t x(t,s) = x$, $x_0(s) = \cos(s)$ $\quad t=0$

$\frac{d}{dt} f = f$

$\frac{df}{f} = dt \xrightarrow{\text{integral}} \ln f = t + C$

$f = e^{t+C} = e^t \cdot K$

$\cos(s) = e^0 \cdot K = K$

(2)
$$\begin{aligned} x(t,s) &= e^{t} \cdot \cos(s) \Rightarrow x^2 = e^{2t}\cos^2(s) \\ y(t,s) &= e^t \cdot \sin(s) \Rightarrow y^2 = e^{2t}\sin^2(s) \\ u(t,s) &= e^{2t} = \text{something with } x \text{ and } y \end{aligned}$$

Add

$x^2+y^2 = e^{2t}\underbrace{(\cos^2 s + \sin^2 s)}_{1}$

$= e^{2t}$

$= u$

$u = x^2+y^2$

1/ $1\cdot\partial_x u + \partial_y u = 1$, $\boxed{u(x,0) = \sin x}$

Step 1: $a = 1$, $b = 1$, $c = 1$

Step 2: Parametrize (Initial):

$x_0(s) = s$, $y_0(s) = 0$, $u_0(s) = \sin s$

Step 3:

$$\begin{cases} \partial_t x = 1, \quad x_0(s) = s & (1) \\ \partial_t y = 1, \quad y_0(s) = 0 & (2) \\ \partial_t u = 1, \quad u_0(s) = \sin(s) & (3) \end{cases}$$

$\partial_t f = 1$: $f = t + K$

$s = x - t$
$= x - y$

(1): $x(t,s) = t + s \rightsquigarrow x = t + s$

(2): $y(t,s) = t + 0 = t \rightsquigarrow y = t$

(3): $u(t,s) = t + \sin(s)$

$u = y + \sin(x - y)$

2/ $x\,\partial_x u + (x+y)\,\partial_y u = \underline{1}$, $\boxed{u(1,y)=y}$

Step 1: $a = x$, $b = (x+y)$, $c = 1$

Step 2: parametrize:

$x_0(s) = 1$, $y_0(s) = s$, $u_0(s) = s$

Step 3:

$$\begin{cases} \partial_t x = x, & x_0(s) = 1 \quad (1) \\ \partial_t y = x+y, & y_0(s) = s \quad (2) \\ \partial_t u = 1, & u_0(s) = s \quad (3) \end{cases}$$

(1): $\boxed{x(t,s) = e^t * 1 = e^t}$ (*) $\Rightarrow$ $t = \ln x$

(3): $u(t,s) = t + s$

(2): $\partial_t y = e^t + y$ | $\frac{d}{dt} f = e^t + f$

$\partial_t y - y = e^t$ | $\frac{d}{dt} f - f = e^t$

$\partial_t (e^{-t} \cdot y) = e^{-t} \cdot (\partial_t y - y)$

$= e^{-t} \cdot e^t = \underline{1}$

integrate on $[0,t]$

$\frac{d}{dt}\left(e^{-t} f\right) = e^{-t} \underbrace{\left(\frac{d}{dt} f - f\right)}$

product rule

$= e^{-t} e^t = \underline{1}$

$e^{-t} y(t,s) - e^{0} \underbrace{y(0,s)} = t$

$e^{-t} y(t,s) - s = t \Rightarrow e^{-t} y = s + t \Rightarrow \boxed{y = e^t (s+t)}$ (**)

(**): $y = x(s + \ln x)$

$\frac{y}{x} = s + \ln x$

$\boxed{s = \frac{y}{x} - \ln x}$

$u = \cancel{\ln x} + \frac{y}{x} - \cancel{\ln x}$

$\boxed{u = \frac{y}{x}}$