* $\underline{\underline{2}}$ * $\quad y\,\partial_x u - u^2\,\partial_y u = 4x \;,\; u(x, x^2) = -2x$

$P(-2, 4, 4)$

Step 1: $a = y \;,\; b = -u^2 \;,\; c = 4x$

Step 2: Initial para:

$x_0(s) = s \;,\; y_0(s) = s^2 \qquad u_0(s) = -2s$

$(s, s^2, -2s) = (-2, 4, 4)$

So $\boxed{s = -2}$

$u_0' = ?$

$a = y_0(-2) = (-2)^2 = 4$

$b = -[\underbrace{u_0}(s)]^2 = -[\underbrace{-2\cdot(-2)}_{4}]^2 = -4^2 = -16$

$x_0' = 1 \rightsquigarrow x_0'(-2) = 1$

$y_0' = 2s \rightsquigarrow y_0'(-2) = 2(-2) = -4$

$$J = \begin{vmatrix} a & b \\ x_0' & y_0' \end{vmatrix} = \begin{vmatrix} 4 & -16 \\ 1 & -4 \end{vmatrix} = 4(-4) - 1(-16) = 0$$

transversality condition fail

- no solution
- infinitely many solution

happens when (a, b, c) is parallel to (x_0', y_0', u_0')

(a, b, c) parallel to $\boxed{(x_0', y_0', u_0')}$

- $a = 4$
  $b = -16$

$s = -2$

$c = 4x = 4s = 4(-2) = -8$

$(a, b, c) = (\underline{4}, \underline{-16}, \underline{-8})$

- $x_0' = 1$
  $y_0' = -4$
  $u_0' = -2$

$(x_0', y_0', u_0') = (1, -4, -2)$

$(a, b, c) = 4 (x_0', y_0', v_0')$

Therefore (a, b, c) is parallel to (x_0', y_0', z_0')

so infinitely many solutions

$\blacksquare$

1 $x\partial_x u - y\partial_y u = -u$ | $P\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 2\right)$

$u(x,y) = 2$ on $\{x^2+y^2=1\}$

Step1: $a = x$, $b = -y$, $c = -u$

Step2: Initial Para:

$x_0(s) = \cos s$, $y_0(s) = \sin s$, $u_0(s) = 2$

$(\cos s, \sin s, 2) = P\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 2\right)$

$$\begin{cases} \cos s = -\frac{\sqrt{2}}{2} \\ \sin s = \frac{\sqrt{2}}{2} \end{cases} \Rightarrow \boxed{s = \frac{3\pi}{4}}$$

* $a = x = \cos s = \cos\left(\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

$b = -y = -\sin s = -\sin\left(\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

* $x' = (\cos s)' = -\sin s = -\sin\left(\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

$y' = (\sin s)' = \cos s = \cos\left(\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

$$J = \begin{vmatrix} a & b \\ x' & y' \end{vmatrix} = \begin{vmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{vmatrix} = 0$$

transversality condition fail < no solution / infinite solution

← not happen!

* Is (a,b,c) and (x_0', y_0', u_0') parallel?

$a = -\frac{\sqrt{2}}{2}$

$b = -\frac{\sqrt{2}}{2}$

$c = -u = -(2) = -2$

$(a,b,c) = \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, -2\right)$

$x_0' = -\frac{\sqrt{2}}{2}$

$y_0' = -\frac{\sqrt{2}}{2}$

$u_0' = (2)' = 0$

$\rightarrow (x_0', y_0', u_0') = \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 0\right)$

Since (a,b,c) and (x_0', y_0', u_0') can never be parallel, so <u>not</u> the case of infinite solution.

So <u>no</u> solution near $P\left(-\frac{\sqrt{2}}{2}, +\frac{\sqrt{2}}{2}, 2\right)$.

* $\underline{\underline{3}}$ * $x^2 \partial_x u + y^2 \partial_y u = u^2$

$u(x, 2x) = -2x \quad , \quad P = (1, 2, -2)$

<u>Step1:</u> $a = x^2 \quad , \quad b = y^2 \quad , \quad c = u^2$

<u>Step2:</u> $x_0(s) = s \quad , \quad y_0(s) = 2s \quad , \quad u_0(s) = -2s$

$$\begin{cases} \partial_t x = x^2, \quad x_0(s) = s & (1) \\ \partial_t y = y^2, \quad y_0(s) = 2s & (2) \\ \partial_t u = u^2, \quad u_0(s) = -2s & (3) \end{cases}$$

$$\frac{dx}{dt} = x^2 \longrightarrow \frac{dx}{x^2} = dt$$

integrate from 0 to t

t

$$-\frac{1}{x(t,s)} - \left(-\frac{1}{x(0,s)}\right) = t$$

$$-\frac{1}{x} + \frac{1}{s} = t \implies \frac{1}{x} = \frac{1}{s} - t = \frac{1-ts}{s}$$

$$y = \frac{2s}{1-2ts} \qquad \boxed{x = \frac{s}{1-ts}} \qquad u = \frac{-2s}{1+2ts}$$

Solve $s = \ldots$, $t = \ldots$ and plug in