HW1 Solution

0 is not an element of U (*)

$$\tau_F = \{ U \subset \mathbb{R} \text{ st } \underbrace{0 \notin U} \text{ or } \underbrace{|\mathbb{R}\setminus U|}< \infty \}$$

number of elements in $\mathbb{R}\setminus U$ is finite (**)

a/ Show τ_F is a topology on $\mathbb{R}$

(i) Show ϕ and $\mathbb{R} \in \tau_F$

- ϕ is in τ_F because 0 is not an element of ϕ
- $\mathbb{R}$ is in τ_F because : $|\mathbb{R}\setminus\mathbb{R}| = |\phi| = 0 < \infty$

(ii) Show finite intersection is also in τ_F.

Let $U_1, U_2, \dots, U_k$ be elements in τ_F.

Case 1: There is at least 1 $i \in \{1, \dots, k\}$ such that $0 \notin U_i$

Then the intersection $U = \bigcap_{j=1}^{k} U_j$ will $\boxed{\text{not}}$ contain 0, hence $U \in \tau_F$ by (*).

Case 2: (negation of case 1): $0 \in U_i \quad \forall i \in \{1, 2, \dots, k\}$

In this case, in order for U_i $(i=1,\dots,k)$ to be elements of τ_F, they must be type $(**)$.

Therefore $|\mathbb{R} \setminus U_i| < \infty$ for all $i=1,\dots,k$

Then denote $\mathcal{U} = \bigcap_{i=1}^{k} U_i$

$$\mathbb{R} \setminus \mathcal{U} = \mathbb{R} \setminus \left(\bigcap_{i=1}^{k} U_i \right) = \bigcup_{i=1}^{k} (\mathbb{R} \setminus U_i)$$

De-Morgan law

complement of intersection

= union of complement

Since $|\mathbb{R} \setminus U_i| < \infty$ for $i=1,\dots,k$

$$\Rightarrow \sum_{i=1}^{k} |\mathbb{R} \setminus U_i| < \infty$$

Thus $|\mathbb{R} \setminus \mathcal{U}| = \left| \bigcup_{i=1}^{k} (\mathbb{R} \setminus U_i) \right|$

$$= \sum_{i=1}^{k} |\mathbb{R} \setminus U_i| < \infty$$

Therefore $\mathcal{U} \in \tau_K$ by $(**)$

(iii) Show arbitrary union is also in τ_K.

Let $\{U_\alpha\}_{\alpha \in I}$ be elements of τ_K

Again we deal with 2 cases: (but quite different! careful!)

Case 1: for all $\alpha \in I$, $0 \notin U_\alpha$.

Then $U = \bigcup_{\alpha \in I} U_\alpha$ does not contain 0

and therefore is an element of τ_K by (*)

Case 2: there is some $\alpha_0 \in I$ st $0 \in U_{\alpha_0}$.

Then in this case, for this particular U_{α_0}, we know

that in order to be in τ_F, $|\mathbb{R} \setminus U_{\alpha_0}| < \infty$

Again, denote $U = \bigcup_{\alpha \in I} U_\alpha$

Then $\mathbb{R} \setminus U = \mathbb{R} \setminus \bigcup_{\alpha \in I} U_\alpha$

$$\underset{\substack{\text{De-Morgan}\\\text{Law}}}{=} \bigcap_{\alpha \in I} (\mathbb{R} \setminus U_\alpha) \subseteq \mathbb{R} \setminus U_{\alpha_0}$$

because $\alpha_0 \in I$

compliment of union = intersection of complement

Then:

$$|\mathbb{R} \setminus U| \underset{\substack{\uparrow \\ \text{since}}}{\leq} |\mathbb{R} \setminus U_{\alpha_0}| < \infty$$

$$\mathbb{R} \setminus U \underset{\text{subset of}}{\subseteq} \mathbb{R} \setminus U_{\alpha_0}$$

So $U \in \tau_F$ by (**)

Now from (i), (ii) and (iii): all axioms are satisfied

Therefore, τ_F is a topology

$\boxtimes$

* Show $\mathbb{R}$ with τ_F is $\boxed{\text{not}}$ connected.

We need to find a separation of $\mathbb{R}$.

Recall:

Defn: X be a topological space. A separation of X is $\boxed{\text{a pair}}$ of U and V non-empty (1), open (2) subset of X st their union is X (3).

Here $X = (\mathbb{R}, \tau_F)$.

Let $U = \{1, 2, 3\}$: non-empty

$V = \mathbb{R} \setminus \{1, 2, 3\}$: non-empty

$\Big\}$ (1) is satisfied

Then U union $V = \mathbb{R}$: so (3) is satisfied

Need to check (2), i.e, are U and V open?

U is open because it does not contain 0.

Why is V open?

$$\mathbb{R} \setminus V = \mathbb{R} \setminus (\mathbb{R} \setminus \{1,2,3\}) = \{1,2,3\}$$

so $|\mathbb{R} \setminus V| = |\{1,2,3\}| = 3 < \infty$

Therefore $V \in \tau_{\mathcal{F}}$ and so V is open.

* So both U and V are open $\Rightarrow$ (2) is satisfied.

Thus U and V is a separation.

* Decide whether $\mathbb{R} = \{0\} \cup (\mathbb{R} \setminus \{0\})$ is a separation of $(\mathbb{R}, \tau_K)$?

Answer: NO because $U = \{0\}$ is **NOT** open.

$|\mathbb{R} \setminus \{0\}| = \infty$

+ Define a continuous surjective map from $(\mathbb{R}, \tau_F)$ to $Y = \{-1, 1\}$ with discrete topology τ_d.

<u>Defn</u> : $f : X \to Y$ is $\boxed{\text{continuous}}$ if preimage of every open set in Y has to be open in X.

What are the open sets in Y with τ_d ?

They are : ϕ , $\underbrace{\{-1, 1\}}_{Y}$, and $\{-1\}$, and $\{1\}$.

Define f by sending : $U = \{1, 2, 3\} \longmapsto \{-1\}$.

$V = \mathbb{R} \setminus \{1, 2, 3\} \longmapsto \{1\}$

Why f continuous?

preimage of $\{-1, 1\}$ = preimage of $\{-1\}$ $\cup$ preimage of $\{1\}$

$= U \cup V = \mathbb{R} \to$ open in τ_K

preimage of ϕ = preimage of $\{-1\}$ $\cap$ preimage of $\{1\}$

$= \phi$: open in τ_K

preimage of $\{-1\} = U$: open in τ_K (not contain 0)

preimage of $\{1\} = V$: open in τ_K (complement is finite)