Math 145B - Discussion W1 Mar 31

Topology

* General Plan:

* Review topology
* Basis
* standard topology on $\mathbb{R}$ ($\mathbb{R}_s$)
* K - topology on $\mathbb{R}$ ($\mathbb{R}_k$)
* difference between $\mathbb{R}_s$ and $\mathbb{R}_k$
* definition of Product Topology

Mar 31, 2022

Week 1

* **Defn:** A topology on a set X, denoted τ, is a collection of subsets of X, satisfying:

1/ arbitrary union of elements of τ must also be an element of τ.

2/ finite intersection of elements of τ must also be an element of τ

3/ ϕ empty set and X are both elements of τ.

In "simpler" words, they are "collections" or rules to tell people what do you mean "open" (or closed) in a set X? Members of τ are the "open" in X. So putting a topology on a set X is telling people what do you mean by open. And of course, different ways of telling people, meaning different "topology" will have different effects. So for example, $\mathbb{R}$, the set of all real numbers can be given many topologies on them. For example the standard topology and the K-topology are quite "different", even though we all give the rules/topology onto the same set $\mathbb{R}$.

I will denote $\mathbb{R}_s$ as $\mathbb{R}$ with standard topology (guess where's the s come from?) and $\mathbb{R}_K$ as $\mathbb{R}$ with the K-topology (guess where's the K-come from? ^^)

* Sometimes (or all the time?) it is too hard (or too lazy) to list all the elements in $\mathcal{T}$,

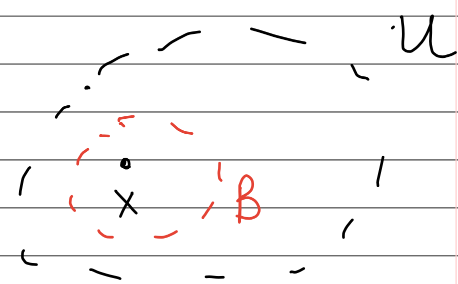
So we come to the basis. In a sense, we can "produce" the entire topology (ie, all elements of it) with just the basis.

How?

Assume we have a $\boxed{\text{basis } \mathcal{B}}$ for $\mathcal{T}$. (with a bunch of $\mathcal{B}$ elements)

A set $U \subset X$ is $\boxed{\text{open}}$, ie, is an element of $\mathcal{T}$

if $\forall x \in U$, we can find a $B \in \mathcal{B}$ such that $x \in B \subset U$. (We can find a B contains x, and B is inside U)



So from now on we just give you basis, then you can find the topology generated by this basis.

Example 1: $\mathcal{B}_s = \{(\text{open intervals in } \mathbb{R})\}$

gives $\mathbb{R}_s$ the standard topology (on $\mathbb{R}$)

Question: Is it really a topology?

This just really going down to the point that is $\mathcal{B}_s$ really a basis? Because if you have a basis, then you generate a topology, Right?

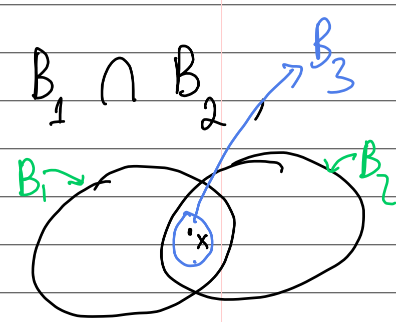
So check that $\mathcal{B}_s$ is a basis, which bring me to the review definition of a basis:

Defn: Let X be a set. $\mathcal{B}$ is a basis (for topology) on X is a collection of subsets in X satisfying:

① $\forall x \in X : \exists B_0 \in \mathcal{B}$ st $x \in B_0$

② If B_1 and $B_2 \in \mathcal{B}$, and $x \in B_1 \cap B_2$

then $\exists B_3 \in \mathcal{B}$ st $x \in B_3 \subset B_1 \cap B_2$.



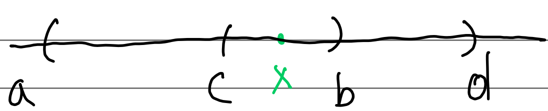
* So is $\mathcal{B} = \{\text{set of all open intervals in } \mathbb{R}\}$ a basis (creating $\mathbb{R}_s$)?

1/ $\forall x \in \mathbb{R}$: $\exists\ \underbrace{(x-1, x+1)}_{\in\, \mathcal{B} \text{ because open interval}}$ st $x \in (x-1, x+1)$

2/ If $\underbrace{(a,b)}$ and $\underbrace{(c,d)}_{}$ are elements in $\mathcal{B}$

open intervals

and let $x \in (a,b) \cap (c,d)$



then $x \in \underbrace{(c,b)}_{\text{open interval}} \in \mathcal{B}$

and $(c,b) \subset (a,b) \cap (c,d)$

So $\mathcal{B}$ is a basis. So $\mathbb{R}_s$ with this basis is a topology!

$\boxtimes$

* Now, recall the K-basis (which gives $\mathbb{R}_K$ topology):

$$K = \left\{ \frac{1}{n} : n \in \mathbb{Z}^+ \right\}$$

$$\mathcal{B}' = \{ \underbrace{(a,b) : a < b}_{*} \} \cup \{ \underbrace{(a,b) - K : a < b}_{**} \}$$

Q: Is $\mathcal{B}'$ a basis? If it is, then this gives the $\mathbb{R}_K$ topology?

A: 1 Let $x \in \mathbb{R}$. Then $(x-1, x+1)$ is an element of $\mathcal{B}'$ (of the form $*$) and containing x. So any $x \in \mathbb{R}$ has an element of $\mathcal{B}'$ containing it: ✓ Yay.

2 Case 1: $x \in * \cap *$

$(a,b) \cap (c,d)$

Then $x \in (c,b) \subset (a,b) \cap (c,d)$

Case 2: $x \in ** \cap *$

$(a,b) \cap (c,d) \setminus K$

In this case $x \in (c,b) \setminus K$

which is of (**) form

Case 3: $x \in$ (**) $\cap$ (**)

$(a,b) \setminus K \cap (c,d) \setminus K$

In this case $x \in \underbrace{(c,b)}_{} \setminus K$ as well

of the form (**)

Therefore β' is a basis

Hence $\mathbb{R}_K$ is a topology $\boxed{}$

Q: What's the difference between $\mathbb{R}_S$ and $\mathbb{R}_K$?

Answer: $\mathbb{R}_K$ is finer than $\mathbb{R}_S$

Lemma: $\underset{\mathcal{B}'}{\mathcal{T}'} \supset \underset{\mathcal{B}}{\mathcal{T}}$, i.e. $\mathcal{T}'$ is finer than $\mathcal{T}$
basis

iff given $x \in X$ and $B \in \mathcal{B}$ st $x \in B$, $\exists\, B' \in \mathcal{B}'$

st $x \in B' \subset B$.

In words, finer if you can find a "basis" element of the finer basis INSIDE the other one.

(I hope this is less confusing!)

Show $\mathbb{R}_K$ is finer than $\mathbb{R}_S$?

Let $x \in \mathbb{R}$ then $x \in \underbrace{(x-1, x+1)}$

B is a basis element of $\mathbb{R}_S$

But B is also a basis element (of form (*))

of $\mathbb{R}_K$. So $x \in B = B$. Done $\boxed{}$

basis element in the finer basis

Question: Are there any set of $\mathbb{R}$ open in $\mathbb{R}_K$ but not open in $\mathbb{R}_S$?

Ans: absolutely yes!

. Take $U = \mathbb{R} \setminus K$. then U is $\boxed{\text{open in } \mathbb{R}_K}$ because U can be written in $\boxed{\text{union}}$ of elements of form $(**)$ in the basis β' of $\mathbb{R}_K$.

. U is NOT open in $\mathbb{R}_S$: since :

$0 \in U$ (the number zero is in U)

but you cannot find any open interval that contains the number zero and is a subset of U.

In order to be a subset of $U = \mathbb{R} \setminus K$, you $\boxed{\text{cannot}}$ contains any fraction of the form $\frac{1}{m}$ (like $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$)

but any open interval around zero must contain at least one such fraction, no matter how small the interval is!

Say $(0, 0.000001)$ still contain $\frac{1}{10^7}$

$\square$

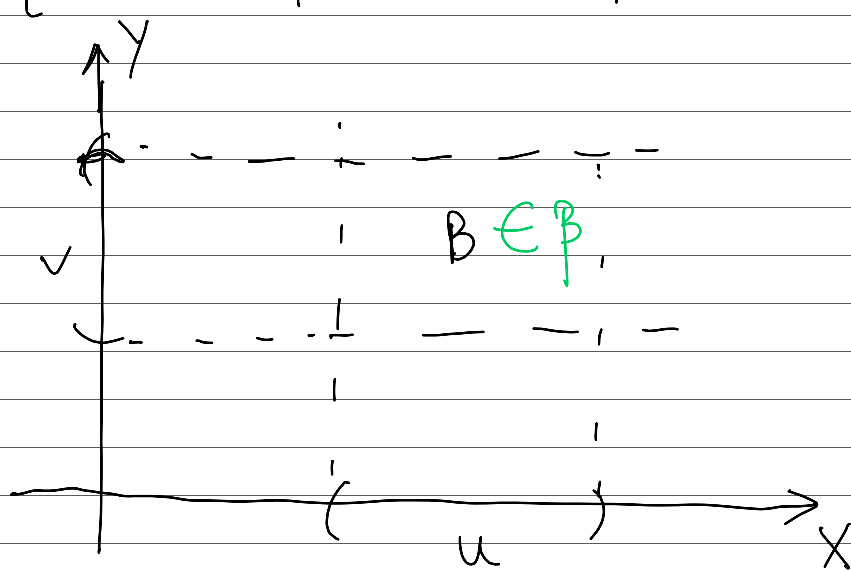
Next we will review product topology!

* ## Defn : Product topology

This definition is given based on a $\boxed{\text{basis}}$. So in order for this to actually be a topology, We must verify that $\mathcal{B}$ is indeed a basis!

Let (X, τ_X) and (Y, τ_Y) be 2 topological spaces. The $\boxed{\text{product topology}}$ on $X \times Y$ is the topology generated by this basis:

$$\mathcal{B} = \{ U \times V \mid U \in \tau_X \text{ and } V \in \tau_Y \}$$



* Verify β is a basis:

(1) Let $(x,y) \in X \times Y$.

X is in τ_X, Y is in τ_Y

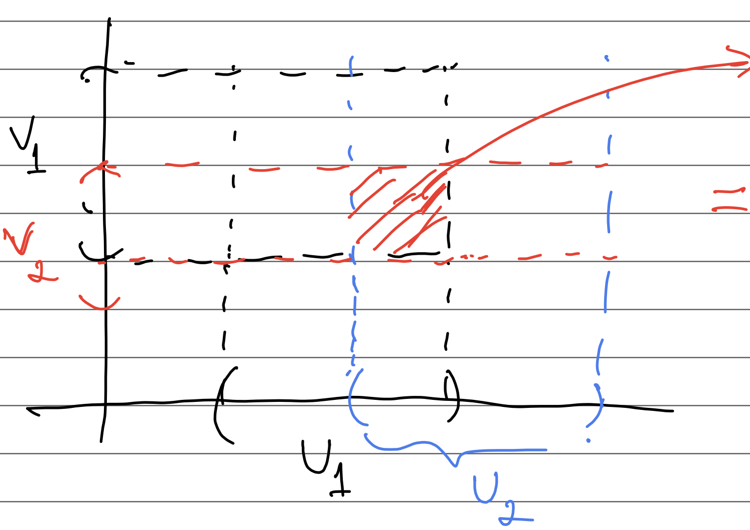
so $X \times Y$ is an element of β.

and $(x,y) \in X \times Y$.

(2) Let $U_1 \times V_1$ and $U_2 \times V_2$ be basis elements

st $(x,y) \in (U_1 \times V_1) \cap (U_2 \times V_2)$

Fact: $(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \times V_2)$



$(U_1 \times V_1) \cap (U_2 \times V_2)$

$= (U_1 \cap U_2) \times (V_1 \times V_2)$

From the fact: $x \in \underbrace{(U_1 \cap U_2)}_{\in \tau_X} \times \underbrace{(V_1 \cap V_2)}_{\in \tau_Y}$ is an element of β