HW3

*** 1 ***

(i) Why X is homotopic to X

Here take $f = g = id_X : X \to X$

Then $f \circ g = g \circ f = id_X : X \to X \sim id_X$

(ii) Let X is homotopic to Y.

So we can find $f : X \to Y$ and $g : Y \to X$

st $f \circ g \sim id_Y$ and $g \circ f \sim id_X$

But this also mean Y homotopic to X

$(g \circ f \sim id_X$ and $f \circ g \sim id_Y)$

(iii) Let X homotopic to Y (1)

Y homotopic to Z (2)

(1) gives 2 functions $f : X \to Y$ and $g : Y \to X$

st $f \circ g \sim id_Y$ and $g \circ f \sim id_X$

(2) gives 2 functions $h : Y \to Z$ and $k : Z \to Y$

st $h \circ k \sim id_Z$ and $k \circ h \sim id_Y$

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* 1* (ctd)

<u>Claim:</u> $h \circ f : X \to Z$ and $g \circ k : Z \to X$

satisfies: $(h \circ f) \circ (g \circ k) \sim id_Z$ (**)

and $(g \circ k) \circ (h \circ f) \sim id_X$ (*)

Show (*): $(g \circ k) \circ (h \circ f) = g \circ \underbrace{(k \circ h)} \circ f$ (associative)

$\sim g \circ \underbrace{id_Y \circ f}$ ($k \circ h \sim id_Y$)

$\sim g \circ f$ ($g \circ f \sim id_X$)

$\sim id_X$

(**): $(h \circ f) \circ (g \circ k) = h \circ (f \circ g) \circ k$

$\sim h \circ \underbrace{(id_X)} \circ k$

$\sim h \circ k \sim id_Y$

2 a/ $\boxed{\Rightarrow}$ Let X be contractible.

Then X is homotopy to $Y = \{a\}$ (only $\underline{1}$ element a)

So $\exists\, f : X \to \{a\}$ and $g : \{a\} \to X$ st:

$\boxed{g \circ f \sim \underline{id}_X}$ and $f \circ g \sim id_{\{a\}}$

so id_X is homotopic to $g \circ f$.

Is $g \circ f$ a constant function? Yes.

$$g \circ f(x) = g(f(x)) = g(\{a\}) = \text{const} \; (\underset{\underline{\underline{\text{const}}}}{g(a) = b})$$

So id_X homotopic to constant fnc.

$\boxed{\Leftarrow}$ Suppose $id_X : X \to X$ identity function is homotopic to a constant function $C_p : X \to X$

(ie, $C_p(x) = p \;\; \forall x \in X$)

$id_X \sim C_p$. (*)

Then $\exists\, H : X \times [0,1] \to X$ with

$$H(x,0) = id_X(x) = x \quad \forall x \in X$$

$$H(x,1) = C_p(x) = p \quad \forall x \in X$$

Now we need to find $f : X \to \{p\}$ and $g : \{p\} \to X$

st $f \circ g \sim id_X$ and $g \circ f \sim id_{\{p\}}$

Existence of these f and g will show X is homotopy equivalence to $\{p\}$.

In fact, take $f = C_p : X \to \{p\}$

$g = id_X : \{p\} \to \{p\}$

* $g \circ f(x) = g(f(x)) = g(p) = p \quad \forall x \in X$

So $g \circ f(x) = C_p(x)$. By (*) earlier $C_p \sim id_X$

homotopy symbol

$\Rightarrow g \circ f \sim id_X$

* $f \circ g(p) = f(p) = C_p(p) = p$

only $p \in \{p\}$

So $f \circ g \boxed{=} id_{\{p\}}$ ← the space has only $\underline{1}$ element

equal sign

$\Rightarrow f \circ g \sim id_{\{p\}}$

So f, g found. $\Rightarrow$ X is homo equivalent to $\{p\}$ $\square$

* 2* b/ Y contractible $\Rightarrow id_Y \sim C_p$ where $C_p(y) = p$ $\forall y \in Y$

Let $f : X \to Y$

Since $id_Y \sim C_p$, there is $H : Y \times [0,1] \to Y$ with

$H(y,0) = id_Y$

$H(y,1) = C_p$.

is in Y

Define $K : X \times [0,1] \to Y$ by $K(x,t) = H(\tilde{f(x)}, t))$

Then: $K(x,0) = H(f(x),0) = id_Y(\underbrace{f(x)}_{\text{in } Y}) = f(x)$

$K(x,1) = H(f(x),1) = C_p(f(x)) = p = C_p$

Existence of K gives $f(x) \sim C_p$.

So f is homotopic. f is arbitrary map $X \to Y$

$\Rightarrow$ all map from $X \to Y$ homotopic

*2*c/ * Show $\underbrace{[0,1]}_{X}$ on $\mathbb{R}_{std}$ contractible:

Define $H: X \times [0,1] \longrightarrow X$ by

$$H(x,t) = xt$$

constant map $C_0(x) = 0 \ \forall x \in X$

H is continuous, and: $H(x,0) = 0 = C_0$

$$H(x,1) = x \cdot 1 = x = id_X(x)$$

So $id_X \sim C_0 \underset{\text{part a}}{\Longrightarrow}$ X is contractible.

* Show: $\mathbb{R}_l$ on $[0,1] \subseteq \mathbb{R}_l$ NOT contractible.

Claim: X is contractible $\Rightarrow$ X is path-connected

proof of Claim: X contractible, so $id_X \sim C_p$

some constant map $C_p : X \to X$ with $C_p(x) = p \ \forall x \in X$

So there is H continuous with $H: X \times [0,1] \to X$:

$$H(x,0) = id_X(x) = x \quad \forall x \in X$$

$$H(x,1) = C_p(x) = p \quad \forall x \in X$$

Let a, b be 2 points in X.

Define $\gamma_a : [0,1] \longrightarrow X$ by $\gamma_a(t) = H(a,t)$

So $\gamma_a(0) = H(a,0) = id_X(a) = a$

$\gamma_a(1) = H(a,1) = C_p(a) = p.$

So γ_a is a path connect a and p.

Define $\gamma_b : [0,1] \longrightarrow X$ by:

"inverse"

$$\gamma_b(t) = H(b, 1-t)$$

Then $\gamma_b(\underline{0}) = H(b, \underbrace{1}_{1-0=1}) = C_p(b) = \underline{p}$

$\gamma_b(\underline{1}) = H(b, \underbrace{0}_{1-1=0}) = id_{\cancel{X}}(b) = \underline{b}$

So γ_b connect p to b

Thus $\gamma_b \circ \gamma_a$ will be a path from a to b.
So X is path-connected.

$\mathbb{R}_\ell$ and $[0,1] \subseteq \mathbb{R}_\ell$ is $\underline{\underline{\text{NOT}}}$ path connected

$\Rightarrow$ not contractible

$\square$