HW4

* (1) * a/ $k : Y \to Z$ map

$f_0, f_1 : X \to Y$ are homotopic.

So there exists $F : X \times [0,1] \to Y$ st:

$F(x,0) = f_0(x)$

$F(x,1) = f_1(x)$

Define $G : X \times [0,1] \to Z$ by:

$$G = k \circ F$$

$G(x,0) = k(F(x,0)) = k(f_0(x)) = k \circ f_0(x)$

↕ 2 functions

$G(x,1) = k(F(x,1)) = k(f_1(x)) = k \circ f_1(x)$

So this G exists $\Rightarrow$ $k \circ f_0 \sim k \circ f_1$

☒

* 1* b/ $k : Y \to Z$ is a map

Let $f_0, f_1 : I_s \longrightarrow Y$ path homotopic

WTS: $k \circ f_0,\ k \circ f_1 : I_s \to Z$ are also path homotopic

Because f_0, f_1 path homotopic

$\exists F : I_s \times I_t \longrightarrow Y$ st:

$F(s,0) = f_0(s)$

$F(s,1) = f_1(s)$

2 "functions"

$F(0,t) = y_0$

$F(1,t) = y_1$

points in Y

Define $G : I_s \times I_t \to Z$ by

$G = k \circ F = k(F)$

compose of function

$G(s,0) = k(F(s,0))$

$= k(f_0(s)) = k \circ f_0(s)$

2 functions

$G(s,1) = k(F(s,1))$

$= k(f_1(s)) = k \circ f_1(s)$

$G(0,t) = k(F(0,t))$

$= k(y_0) = z_0$: points in Z

$G(1,t) = k(F(1,t))$

$= k(y_1) = z_1$

The existence of $G : I_s \times I_t \to Z$

show $k \circ f_0$ path homo to $k \circ f_1$ $\boxed{\text{Ma}}$

2 a/

+ First consider the case of taking out the north pole $N = (0, 0, \dots, 0, 1) \in \mathbb{R}^{n+1}$

We want to find $f: S^n \setminus \{N\} \longrightarrow \mathbb{R}^n$

st f is continuous, bijective. (f is your homeomorphic function)

Define $f(\vec{x}) = f(x_0, \dots, x_1, x_{n+1}) = \dfrac{1}{1 - x_{n+1}} (x_0, \dots, x_n)$

Now define:

$g: \mathbb{R}^n \longrightarrow S^n \setminus \{N\}$ by:

$$\vec{u} = (u_1, \dots, u_n) \xmapsto[\text{sends to}]{} \frac{1}{\|\vec{u}\|^2 - 1} (2u_1, 2u_2, \dots, 2u_n, \underbrace{\|\vec{u}\|^2 + 1}_{\text{last coordinate}})$$

We will show g is inverse of f, (therefore f is bijective)

by showing: $g \circ f = \text{id}$, meaning $\forall \boxed{\vec{x} \in S^n \setminus \{N\}}$

$$g \circ f(\vec{x}) = g(f(\vec{x})) = \vec{x} \quad (*)$$

④

Let $\vec{x} \in S^n \setminus \{N\}$ with $\vec{x} = (x_1, x_2, \ldots, x_n, x_{n+1})$

Since $\vec{x} \in S^n$, so: $\|\vec{x}\|^2 = 1$

Important

So: $x_1^2 + x_2^2 + \cdots + x_n^2 + x_{n+1}^2 = 1$ $(**)$

Now:

$$g(f(\vec{x})) = g\left(\underbrace{\frac{x_1}{1-x_{n+1}}}_{u_1}, \underbrace{\frac{x_2}{1-x_{n+1}}}_{u_2}, \ldots, \underbrace{\frac{x_n}{1-x_{n+1}}}_{u_n}\right)$$

Name $u_i = \dfrac{x_i}{1-x_{n+1}}$ where $i = 1, 2, \ldots, n$

Compute $g(u_1, \ldots, u_n) = \dfrac{1}{\|\vec{u}\|^2+1}\left(2u_1, \ldots, 2u_n, \|\vec{u}\|^2 - 1\right)$

see red part

See the green part below (next page)

• $\|\vec{u}\|^2 = u_1^2 + u_2^2 + \ldots + u_n^2$

$$= \frac{x_1^2}{(1-x_{n+1})^2} + \frac{x_2^2}{(1-x_{n+1})^2} + \cdots + \frac{x_n^2}{(1-x_{n+1})^2}$$

$$= \frac{x_1^2 + \cdots + x_n^2}{(1-x_{n+1})^2}$$

$$\|\vec{u}\|^2 - 1 = \frac{x_1^2 + x_2^2 + \dots + x_n^2}{(1 - x_{n+1})^2} - 1$$

$$= \frac{x_1^2 + \dots + x_n^2 - (1 - x_{n+1})^2}{(1 - x_{n+1})^2}$$

now use (**)

$$= \frac{x_1^2 + \dots + x_n^2 - 1 + 2x_{n+1} - x_{n+1}^2}{(1 - x_{n+1})^2}$$

$$= \frac{\cancel{x_1^2} + \dots + \cancel{x_n^2} - (\cancel{x_1^2} + \cancel{x_2^2} + \dots + \cancel{x_n^2} + x_{n+1}^2) + 2x_{n+1} - x_{n+1}^2}{(1 - x_{n+1})^2}$$

$$= \frac{2x_{n+1} - 2x_{n+1}^2}{(1 - x_{n+1})^2} = \frac{2x_{n+1}}{1 - x_{n+1}}$$

- $\|\vec{u}\|^2 + 1 = \frac{x_1^2 + x_2^2 + \dots + x_n^2}{(1 - x_{n+1})^2} + 1$

$$= \frac{x_1^2 + x_2^2 + \dots + x_n^2 + (1 - x_{n+1})^2}{(1 - x_{n+1})^2}$$

use (**)

$$\|\vec{u}\|^2 + 1 = \frac{x_1^2 + \dots + x_n^2 + 1 - 2x_{n+1} + x_{n+1}^2}{(1 - x_{n+1})^2}$$

$$= \frac{(x_1^2 + \dots + x_n^2 + x_{n+1}^2) + 1 - 2x_{n+1}}{(1 - x_{n+1})^2}$$

$$= \frac{1 + 1 - 2x_{n+1}}{(1 - x_{n+1})^2} = \frac{2 - 2x_{n+1}}{(1 - x_{n+1})^2}$$

$$= \frac{2}{1 - x_{n+1}}$$

Now putting everything:

$$g(u_1, \dots, u_n) = \frac{1}{\left(\frac{2}{1 - x_{n+1}}\right)} \left(\underbrace{\frac{2x_1}{1 - x_{n+1}}}_{u_1}, \dots, \underbrace{\frac{2x_n}{1 - x_{n+1}}}_{u_n}, \underbrace{\frac{2x_{n+1}}{1 - x_{n+1}}}_{\|\vec{u}\|^2 - 1} \right)$$

$$= \frac{1 - x_{n+1}}{2} \left(\frac{2x_1}{1 - x_{n+1}}, \dots, \frac{2x_n}{1 - x_{n+1}}, \frac{2x_{n+1}}{1 - x_{n+1}} \right)$$

$$= (x_1, \dots, x_n, x_{n+1})$$

So $g(u_1, \dots, u_n) = (x_1, \dots, x_n, x_{n+1})$

$$\|$$

$$g \circ f(x_1, \dots, x_n, x_{n+1}) = (x_1, \dots, x_n, x_{n+1})$$

$$g \circ f(\vec{x}) = \vec{x}$$

True $\forall \vec{x} \in S^n \setminus \{N\}$. Therefore bijective of f is shown.

* Now, for any $p \in S^n$, $p \neq N$, the function $h : S^n \setminus \{N\} \to S^n \setminus \{p\}$ defined below is continuous and bijective, so is homeomorphic : $h(\vec{x}) = \vec{x} \quad \forall \vec{x} \neq p$

$$h(\vec{p}) = N$$

The inverse of h is $k : S^n \setminus \{p\} \to S^n \setminus \{N\}$ by

$$\begin{cases} k(\vec{x}) = \vec{x} & \forall \vec{x} \neq N \\ k(N) = \vec{p} \end{cases}$$

Note that because h has inverse k then h is bijective.

Finally, the homeomorphic from $S^n \setminus \{p\}$ to $\mathbb{R}^n$ is: $f \circ h : S^n \setminus \{p\} \longrightarrow \mathbb{R}^n$

$$\left(\begin{array}{l} h : S^n \setminus \{p\} \to S^n \setminus \{N\} \\ f : S^n \setminus \{N\} \to \mathbb{R}^n \end{array} \right)$$

The composition of homeomorphic maps is homeomorphic, so $f \circ h$ is homeomorphic

$\square$

* 2 b * $f : X \to S^n$ not surjective.

So $\exists p \in S^n$ st $p \notin im(f)$ $(= range(f))$

From part a, let $\boxed{g : S^n \setminus \{p\} \to \mathbb{R}^n}$ be the homeomorphic map "connects" $S^n \setminus \{p\}$ to $\mathbb{R}^n$.

Now $g \circ f : X \to \mathbb{R}^n$

Since $\mathbb{R}^n$ is contractible, $g \circ f \sim C_q$ for some constant q and C_q is the constant map (HW3)

In particular there is $H : X \times [0,1] \to \mathbb{R}^n$ by:

$$H(x, 0) = g \circ f(x)$$
$$H(x, 1) = C_q(x)$$

this is the homotopy $f \sim C_t$

Since g is homeomorphic, call h to be g inverse. So $\underbrace{h = g^{-1}}_{h \circ g = id}$. Define K: $X \times [0,1] \to \mathbb{R}^n$ by

$$K = h \circ H$$

$$K(x,0) = h(H(x,0)) = \boxed{h \circ g}(f(x)) = \boxed{f(x)}$$
$$K(x,1) = h(H(x,1)) = h \circ (C_q(x)) = h \circ (q) = \boxed{t} \in S^n$$

const since q is const