SAMPLE
Final Exam
MATH 009A

Exam is administered during Finals Week and is cumulative.

This sample exam covers the following sections of the textbook:

- (1.3) Finding Limits Analytically
- (1.4) One-Sided Limits
- (1.5) Continuity
- (1.6) Limits Involving Infinity
- (2.1) Instantaneous Rates of Change: The Derivative
- (2.2) Interpretations of the Derivative
- (2.3) Basic Differentiation Rules
- (2.4) The Product and Quotient Rules
- (2.5) The Chain Rule
- (2.6) Implicit Differentiation
- (2.7) Derivatives of Inverse Functions
- (6.7) L'Hopital's Rule
- (3.1) Extreme Values
- (3.2) The Mean Value Theorem
- (3.3) Increasing and Decreasing Functions
- (3.4) Concavity and the Second Derivative
- (3.5) Curve Sketching
- (4.2) Related Rates

SAMPLE - Final Exam - MATH 009A

Pink text indicates textbook sections. These will not be listed on the Final exam.

1. Evaluate the following limits.

(a) (1.3) $\lim_{x \to 8} f(x)$, given that $\lim_{x \to 8} \frac{xf(x)}{3} = -2$

(b) (1.3) $\lim_{x \to -3} \frac{x^3 - 9x}{2x + 6}$

(c) (1.3) $\lim_{x \to 0} \frac{\sin^2(3x)}{x}$

2. Evaluate the following limits.

(a) (1.4) $\lim_{x \to 1^-} f(x)$, given that $f(x) = \begin{cases} -x + 2 & x < 1 \\ 3 & 1 \leq x \leq 4 \\ (x-4)^2 & x > 4 \end{cases}$

(b) (1.6) $\lim_{x \to \infty} \frac{2x^2 - 3\sqrt{x} + 5}{x^4 - 2x + 1}$

(c) (6.7) $\lim_{x \to \infty} \frac{x^3}{\ln(x)}$

3. (1.3) Apply the Squeeze/Sandwich Theorem to evaluate $\lim_{x \to 0} x^2 \cos\left(\frac{2}{x}\right)$.

4. (2.1) Find all values of c that make the following function continuous.

$$f(x) = \begin{cases} cx^2 + 2x & x \leq 2 \\ x^3 - cx & x > 2 \end{cases}$$

5. Let $f(x) = 3x - x^2$.

(a) (2.1) Find $f'(x)$ using the limit definition of the derivative.

(b) (2.3) Find $f'(x)$ using the Power Rule.

Verify that your final answer to Part (b) is equal to your final answer to Part (a).

6. Find the derivative of each function below. **You do not need to simplify your answers.**

(a) (2.2, 2.4, 2.3) $f(x) = \sin(x)\ln(x) - \sqrt{\pi^7}$

(b) (2.4, 2.3) $g(x) = \frac{x^2 - x}{2^x + 1}$

7. Find the derivative of each function below. **You do <u>not</u> need to simplify your answers.**

(a) (2.5, 2.7, 2.3) $F(x) = \arcsin(2e^x)$

(b) (2.4, 2.5, 2.3) $G(x) = x\cos\left(\dfrac{x^2+3}{x^2-1}\right)$

8. Let $f(x) = x^3 - 6x^2 + 5$.

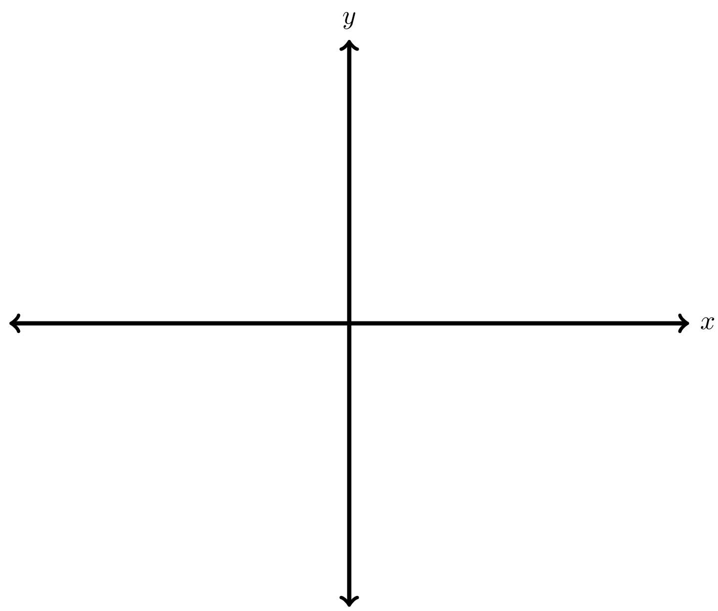
(a) (3.3) Determine where f is increasing and where it is decreasing.

(b) (3.3) Find all local maximums and local minimums of f. Write each point as an ordered pair (x, y).

(c) (3.4) Determine where f is concave up and where it is concave down.

(d) (3.4) Find all points of inflection of f. Write each point as an ordered pair (x, y).

(e) (3.5) Sketch the graph of f on the axes provided below. Clearly label all maximums, minimums, and points of inflection.



(f) (3.1) Find the global maximum and global minimum of f on $[-3, 5]$, as guaranteed by the Extreme Value Theorem. Write each point as an ordered pair (x, y).

(g) (3.2) Find a value of c that satisfies the conclusion of the Mean Value Theorem for f on $[-3, 3]$.

9. (2.6, 4.2) A lighthouse is located on a small island 3 kilometers (km) away from the nearest point P on a straight shoreline. Its light makes 4 revolutions per minute. How fast is the beam of light moving along the shoreline when it is shining on a point that is 1 km away from P? Include units.

10. (3.3) Let f be a function whose derivative is given by the formula $f'(x) = e^{-2x}(3-x)(x+1)^2$. Determine all critical numbers of f and decide whether a relative maximum, relative minimum, or neither occurs at each.

Graphical Problems:

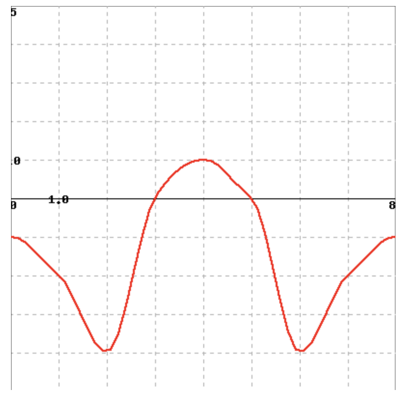
11. (3.3, 3.4) Below is the graph of the derivative $f'(x)$ of a function defined on the interval $(0, 8)$.

(a) For what values of x in $(0, 8)$ is $f(x)$ increasing? (Use interval notation.)

(b) For what values of x in $(0, 8)$ is $f(x)$ concave down? (Use interval notation.)

(c) Find all values of x in $(0, 8)$ where $f(x)$ has a local minimum.

(d) Find all values of x in $(0, 8)$ where $f(x)$ has an inflection point.



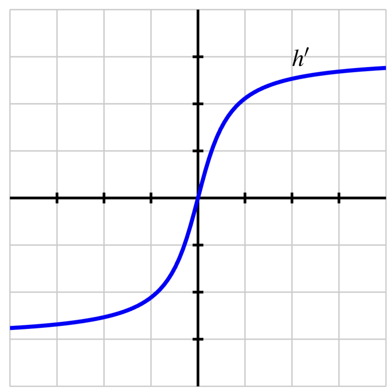
12. (3.3, 3.4) Suppose h is a differentiable function whose first derivative is given by the graph below.

(a) How many real number solutions can the equation $h(x) = 0$ have? Why?

(b) If $h(x) = 0$ has two distinct real solutions, what can you say about the signs of the two solutions (positive vs negative)? Why?

(c) Assume that $\lim_{x\to\infty} h'(x) = 3$, as appears to be indicated on the graph. How will the graph of $y = h(x)$ appear as $x \to \infty$? Why?

(d) Describe the concavity of $y = h(x)$ as fully as you can from the provided information.



Worksheets and midterms!

13. Worksheet 1, Problem 4

14. Worksheet 2, problem 2, 4

15. Worksheet 5, Problem 4

16. Worksheet 6, Problem 1, 2

17. Worksheet 8, Problem 2

18. Worksheet 9, problem 3

19. All problems from the midterms and practice exams!

SOLUTIONS

1. Evaluate the following limits.

(a) (1.3) $\lim_{x\to 8} f(x)$, given that $\lim_{x\to 8} \dfrac{xf(x)}{3} = -2$

Solution:

Applying properties of limits, we get

$$\lim_{x\to 8} \frac{xf(x)}{3} = -2 \implies \frac{\left(\lim_{x\to 8} x\right)\cdot\left(\lim_{x\to 8} f(x)\right)}{\lim_{x\to 8} 3} = -2$$
$$\implies \frac{8\cdot \lim_{x\to 8} f(x)}{3} = -2$$
$$\implies \lim_{x\to 8} f(x) = -2\left(\frac{3}{8}\right) = \boxed{-\frac{3}{4}}$$

(b) (1.3) $\lim_{x\to -3} \dfrac{x^3-9x}{2x+6}$

Solution:

$$\frac{(-3)^3-9(-3)}{2(-3)+6} = \frac{-27+27}{-6+6} = \frac{0}{0}$$

Factor the numerator and denominator:

$$\frac{x^3-9x}{2x+6} = \frac{x(x^2-9)}{2(x+3)} = \frac{x(x+3)(x-3)}{2(x+3)} = \frac{x(x-3)}{2}$$

$$\implies \lim_{x\to -3} \frac{x^3-9x}{2x+6} = \lim_{x\to -3} \frac{x(x-3)}{2} = \frac{(-3)(-6)}{2} = \boxed{9}$$

(c) (1.3) $\lim_{x\to 0} \dfrac{\sin^2(3x)}{x}$

Solution:

We know that for any real number a, $\lim_{x\to 0} \dfrac{\sin(ax)}{ax} = 1$.

We have $\dfrac{\sin^2(3x)}{x}$, but we want $\dfrac{\sin(3x)}{3x}$ so we can use the result above.

Rewrite: $\dfrac{\sin^2(3x)}{x} = \sin(3x)\cdot\dfrac{\sin(3x)}{x} = \sin(3x)\cdot\dfrac{\sin(3x)}{x}\cdot\left(\dfrac{3}{3}\right) = 3\sin(3x)\cdot\dfrac{\sin(3x)}{3x}$

$$\implies \lim_{x\to 0}\left(3\sin(3x)\cdot\frac{\sin(3x)}{3x}\right) = \left(\lim_{x\to 0} 3\sin(3x)\right)\cdot\left(\lim_{x\to 0}\frac{\sin(3x)}{3x}\right) = 0\cdot 1 = \boxed{0}$$

2. Evaluate the following limits.

(a) (1.4) $\lim_{x\to 1^-} f(x)$, given that $f(x) = \begin{cases} -x+2 & x < 1 \\ 3 & 1 \le x \le 4 \\ (x-4)^2 & x > 4 \end{cases}$

Solution:

To the left of $x = 1$, $f(x) = -x + 2$.

So $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (-x+2) = -1 + 2 = \boxed{1}$

(b) (1.6) $\lim_{x\to\infty} \dfrac{2x^2 - 3\sqrt{x} + 5}{x^4 - 2x + 1}$

Solution:

The dominant term in the numerator is $2x^2$, and the dominant term in the denominator is x^4.

$$\implies \lim_{x\to\infty} \frac{2x^2 - 3\sqrt{x} + 5}{x^4 - 2x + 1} = \lim_{x\to\infty} \frac{2x^2}{x^4} = \lim_{x\to\infty} \frac{2}{x^2} = \boxed{0}$$

(c) (6.7) $\lim_{x\to\infty} \dfrac{x^3}{\ln(x)}$

Solution:

$\lim_{x\to\infty} \dfrac{x^3}{\ln(x)}$ has the form $\dfrac{\infty}{\infty}$

By L'Hopital's Rule, $\lim_{x\to\infty} \dfrac{x^3}{\ln(x)} = \lim_{x\to\infty} \dfrac{\frac{d}{dx}x^3}{\frac{d}{dx}\ln(x)} = \lim_{x\to\infty} \dfrac{3x^2}{1/x} = \lim_{x\to\infty} 3x^3 = \boxed{\infty}$

3. (1.3) Apply the Squeeze/Sandwich Theorem to evaluate $\lim_{x\to 0} x^2 \cos\left(\dfrac{2}{x}\right)$.

Solution:

$$\begin{aligned}
-1 \le \cos\left(\frac{2}{x}\right) &\le 1 \quad \text{for all } x \\
\implies -x^2 \le x^2\cos\left(\frac{2}{x}\right) &\le x^2 \\
\implies \lim_{x\to 0}(-x^2) \le \lim_{x\to 0} x^2\cos\left(\frac{2}{x}\right) &\le \lim_{x\to 0} x^2 \\
\implies 0 \le \lim_{x\to 0} x^2\cos\left(\frac{2}{x}\right) \le 0 &\implies \lim_{x\to 0} x^2\cos\left(\frac{2}{x}\right) = \boxed{0}
\end{aligned}$$

4. (2.1) Find all values of c that make the following function continuous.

$$f(x) = \begin{cases} cx^2 + 2x & x \leq 2 \\ x^3 - cx & x > 2 \end{cases}$$

Solution:

For any value of c, $cx^2 + 2x$ and $x^3 - cx$ are continuous since they're polynomials.

But f will not be continuous at $x = 2$ if $\lim_{x \to 2} f(x)$ does not exist (jump discontinuity).

To make f continuous at $x = 2$, we need $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x)$.

$c(2)^2 + 2(2) = (2)^3 - c(2) \implies 4c + 4 = 8 - 2c \implies 6c = 4 \implies c = \frac{4}{6} = \frac{2}{3}$

So f is continuous when $\boxed{c = \frac{2}{3}}$

5. Let $f(x) = 3x - x^2$.

(a) (2.1) Find $f'(x)$ using the limit definition of the derivative.

Solution:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f(x+h) = 3(x+h) - (x+h)^2 = 3x + 3h - (x^2 + 2xh + h^2) = 3x + 3h - x^2 - 2xh - h^2$$

$$\begin{aligned} f(x+h) - f(x) &= (3x + 3h - x^2 - 2xh - h^2) - (3x - x^2) \\ &= \boldsymbol{3x} + 3h \underline{-x^2} - 2xh - h^2 \boldsymbol{-3x} \underline{+x^2} \\ &= 3h - 2xh - h^2 \\ &= h(3 - 2x - h) \end{aligned}$$

$$\frac{f(x+h) - f(x)}{h} = \frac{h(3 - 2x - h)}{h} = 3 - 2x - h$$

$$\implies \lim_{h \to 0} (3 - 2x - h) = 3 - 2x - 0 = \boxed{3 - 2x}$$

(b) (2.3) Find $f'(x)$ using the Power Rule.

Verify that your final answer to Part (b) is equal to your final answer to Part (a).

Solution:

$f'(x) = 3 - 2 \cdot x^{2-1} = \boxed{3 - 2x}$

Yes, my answer to Part (b) is equal to my answer to Part (a).

6. Find the derivative of each function below. **You do <u>not</u> need to simplify your answers.**

(a) (2.4, 2.3) $f(x) = \sin(x)\ln(x) - \sqrt{\pi^7}$

Solution:

Use the sum rule first and the fact that $\sqrt{\pi^7}$ is a constant. Then note that the first term of f is a product, so we use the Product Rule to find its derivative f'.

$$\begin{aligned} f'(x) &= \sin(x) \cdot \frac{d}{dx}\ln(x) + \ln(x) \cdot \frac{d}{dx}\sin(x) - \frac{d}{dx}\sqrt{\pi^7} \\ &= \sin(x) \cdot \frac{1}{x} + \ln(x) \cdot \cos(x) - 0 \\ &= \boxed{\frac{\sin(x)}{x} + \ln(x)\cos(x)} \end{aligned}$$

(b) (2.4, 2.3) $g(x) = \dfrac{x^2 - x}{2^x + 1}$

Solution:

g is a quotient, so we use the Quotient Rule to find its derivative g'.

$$\begin{aligned} g'(x) &= \frac{(2^x + 1) \cdot \frac{d}{dx}(x^2 - x) - (x^2 - x) \cdot \frac{d}{dx}(2^x + 1)}{(2^x + 1)^2} \\ &= \frac{(2^x + 1) \cdot (2x - 1) - (x^2 - x) \cdot (2^x \ln(x) + 0)}{(2^x + 1)^2} \\ &= \boxed{\frac{(2^x + 1)(2x - 1) - (x^2 - x)\big(2^x \ln(x)\big)}{(2^x + 1)^2}} \end{aligned}$$

7. Find the derivative of each function below. **You do <u>not</u> need to simplify your answers.**

(a) (2.5, 2.7, 2.3) $F(x) = \arcsin(2e^x)$

Solution:

F is a composition, so we use the Chain Rule to find its derivative F'.

$\arcsin(2e^x) = \big(\arcsin(x)\big) \circ \big(2e^x\big)$

$$\frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1 - x^2}}$$

$$\frac{d}{dx}2e^x = 2e^x$$

$$\begin{aligned} F'(x) &= \frac{1}{\sqrt{1-(2e^x)^2}} \cdot (2e^x) \\ &= \boxed{\frac{2e^x}{\sqrt{1-4e^{2x}}}} \end{aligned}$$

(b) (2.4, 2.5, 2.3) $G(x) = x\cos\left(\dfrac{x^2+3}{x^2-1}\right)$

Solution:

G is a product, so we use the Product Rule to find its derivative G'.

$$G'(x) = x \cdot \frac{d}{dx}\cos\left(\frac{x^2+3}{x^2-1}\right) + \cos\left(\frac{x^2+3}{x^2-1}\right) \cdot \frac{d}{dx}x$$

$\cos\left(\dfrac{x^2+3}{x^2-1}\right)$ is a composition, so we use the Chain Rule to find its derivative.

$$\cos\left(\frac{x^2+3}{x^2-1}\right) = \big(\cos(x)\big) \circ \left(\frac{x^2+3}{x^2-1}\right)$$

$$\frac{d}{dx}\cos(x) = -\sin(x)$$

$$\begin{aligned} \frac{d}{dx}\left(\frac{x^2+3}{x^2-1}\right) &= \frac{(x^2-1)\cdot \frac{d}{dx}(x^2+3) - (x^2+3)\cdot\frac{d}{dx}(x^2-1)}{(x^2-1)^2} \quad \text{[Quotient Rule]} \\ &= \frac{(x^2-1)(2x)-(x^2+3)(2x)}{(x^2-1)^2} \\ &= \frac{2x\big((x^2-1)-(x^2+3)\big)}{(x^2-1)^2} \\ &= \frac{2x(x^2-1-x^2-3)}{(x^2-1)^2} \\ &= \frac{2x(-4)}{(x^2-1)^2} \\ &= \frac{-8x}{(x^2-1)^2} \end{aligned}$$

$$\implies \frac{d}{dx}\cos\left(\frac{x^2+3}{x^2-1}\right) = -\sin\left(\frac{x^2+3}{x^2-1}\right) \cdot \left(\frac{-8x}{(x^2-1)^2}\right)$$

So $G'(x) = x \cdot \left(-\sin\left(\dfrac{x^2+3}{x^2-1}\right)\cdot\left(\dfrac{-8x}{(x^2-1)^2}\right)\right) + \cos\left(\dfrac{x^2+3}{x^2-1}\right)\cdot(1)$

$$= \boxed{\sin\left(\frac{x^2+3}{x^2-1}\right)\cdot\left(\frac{8x^2}{(x^2-1)^2}\right) + \cos\left(\frac{x^2+3}{x^2-1}\right)}$$

8. Let $f(x) = x^3 - 6x^2 + 5$.

(a) (3.3) Determine where f is increasing and where it is decreasing.

Solution:

f is increasing where $f' > 0$, and f is decreasing where $f' < 0$.

$f'(x) = 3x^2 - 12x$ [Power Rule]

$f'(x) = 0 \implies 3x^2 - 12x = 0 \implies 3x(x-4) = 0 \implies 3x = 0$ or $x - 4 = 0 \implies x = 0$ or $x = 4$

$\underline{\text{On } (-\infty, 0)}$
$f'(-1) = 3(-1)^2 - 12(-1) = 3(1) + 12 = 15 > 0$
$\implies f$ is increasing on $(-\infty, 0)$

$\underline{\text{On } (0, 4)}$
$f'(1) = 3(1)^2 - 12(1) = 3 - 12 = -9 < 0$
$\implies f$ is decreasing on $(0, 4)$

$\underline{\text{On } (4, \infty)}$
$f'(5) = 3(5)^2 - 12(5) = 3(25) - 60 = 75 - 60 = 15 > 0$
$\implies f$ is increasing on $(4, \infty)$

Putting this all together,

$\boxed{f \text{ is increasing on } (-\infty, 0) \cup (4, \infty)}$ and $\boxed{f \text{ is decreasing on } (0, 4)}$

(b) (3.3) Find all local maximums and local minimums of f. Write each point as an ordered pair (x, y).

Solution:

f goes from increasing to decreasing at $x = 0$, so there is a local maximum at $x = 0$.

$f(0) = (0)^3 - 6(0)^2 + 5 = 5$
$\boxed{\text{local maximum at } (0, 5)}$

f goes from decreasing to increasing at $x = 4$, so there is a local minimum at $x = 4$.

$f(4) = (4)^3 - 6(4)^2 + 5 = 64 - 6(16) + 5 = 64 - 96 + 5 = -27$
$\boxed{\text{local minimum at } (4, -27)}$

(c) (3.4) Determine where f is concave up and where it is concave down.

Solution:

f is concave up where $f'' > 0$, and f is concave down where $f'' < 0$.

$f''(x) = 6x - 12$ [Power Rule]

$f''(x) = 0 \implies 6x - 12 = 0 \implies 6x = 12 \implies x = 2$

<u>On $(-\infty, 2)$</u>
$f''(-1) = 6(-1) - 12 = -6 - 12 = -18 < 0$
$\implies f$ is concave down on $(-\infty, 2)$

<u>On $(2, \infty)$</u>
$f''(2) = 6(3) - 12 = 18 - 12 = 6 > 0$
$\implies f$ is concave up on $(2, \infty)$

Putting this all together,

$\boxed{f \text{ is concave down on } (-\infty, 2)}$ and $\boxed{f \text{ is concave up on } (2, \infty)}$

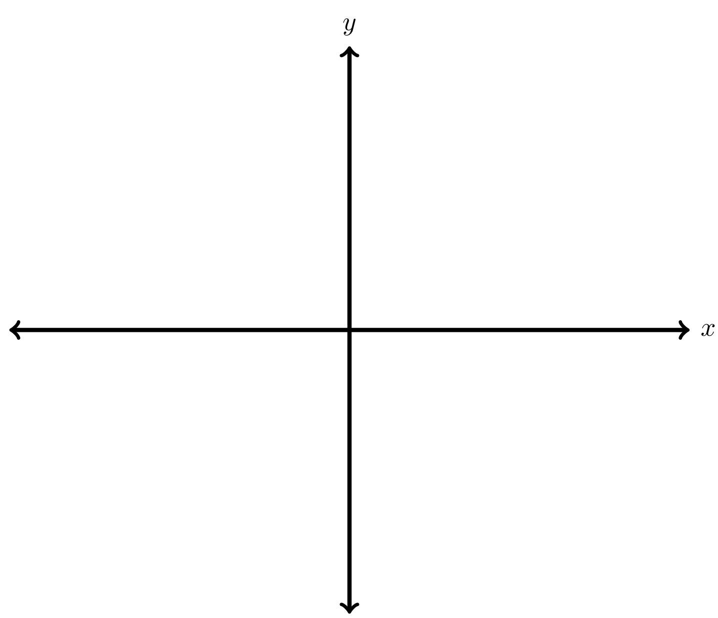
(d) (3.4) Find all points of inflection of f. Write each point as an ordered pair (x, y).

Solution:

f goes from concave down to concave up at $x = 2$, so there is an inflection point at $x = 2$.

$f(2) = (2)^3 - 6(2)^2 + 5 = 8 - 6(4) + 5 = 8 - 24 + 5 = -11$
$\boxed{\text{inflection point at at } (2, -11)}$

(e) (3.5) Sketch the graph of f on the axes provided below. Clearly label all maximums, minimums, and points of inflection.



Solution:

<u>Click here</u> to view the solution on Desmos.

(f) (3.1) Find the global maximum and global minimum of f on $[-3, 5]$, as guaranteed by the Extreme Value Theorem. Write each point as an ordered pair (x, y).

Solution:

On $(-3, 5)$
We know from Part (b) that $(0, 5)$ is a local maximum and $(4, -27)$ is a local minimum.

At $x = -3$ and $x = 5$
$f(-3) = (-3)^3 - 6(-3)^2 + 5 = -27 - 6(9) + 5 = -27 - 54 + 5 = -76$
$f(5) = (5)^3 - 6(5)^2 + 5 = -125 - 6(25) + 5 = 125 - 150 + 5 = -20$

Compare all y-values
If we write the four points $(0, 5)$, $(4, -27)$, $(-3, -76)$, and $(5, -20)$ in ascending order according to y-values, then we get

$$\underbrace{(-3, -76)}_{\text{global min}} \qquad (4, -27) \qquad (5, -20) \qquad \underbrace{(0, 5)}_{\text{global max}}$$

There is a $\boxed{\text{global maximum at } (0, 5)}$ and a $\boxed{\text{global minimum at } (-3, -76)}$.

(g) (3.2) Find a value of c that satisfies the conclusion of the Mean Value Theorem for f on $[-3, 3]$.

Solution:

The average rate of change of f on $[-3, 3]$ is

$\frac{f(3) - f(-3)}{3 - -3} = \frac{-22 - -76}{6} = \frac{54}{6} = 9$
The instantaneous rate of change of f is
$f'(x) = 3x^2 - 12x$
The conclusion of MVT says there is a c on $[-3, 3]$ such that $f'(c) = 9$. So solve the following quadratic quation using the quadratic formula
$3c^2 - 12c = 9 \ 3(c^2 - 4c - 3) = 0$
$c = \frac{4 \pm \sqrt{4^2 - 4(1)(-3)}}{2(1)} = \frac{4 \pm 2\sqrt{7}}{2}$
Since $2 + \sqrt{7} > 4$, it is outside the interval $[-3, 3]$, so $\boxed{c = 2 - \sqrt{7}}$ is the x-value on $(-3, 3)$ such that $f'(c) = \frac{f(3) - f(-3)}{3 - -3}$.

9. (2.6, 4.2) A lighthouse is located on a small island 3 kilometers (km) away from the nearest point P on a straight shoreline. Its light makes 4 revolutions per minute. How fast is the beam of light moving along the shoreline when it is shining on a point that is 1 km away from P? Include units.

Solution:

Click here to view the solution. $\boxed{\frac{80\pi}{3} \text{ km/min}}$