Math 9A Worksheet 4 Solutions

Problem 1

The worksheet gives a graph of $y = f(x)$ and asks about secant lines, tangent lines, average rate of change, and instantaneous rate of change. Since the values are read from a picture, the numerical answers are approximate.

(a) Secant and tangent lines

Secant line on $[-3, -1]$. A secant line is the line through the two points

$$(-3, f(-3)) \quad \text{and} \quad (-1, f(-1)).$$

Secant line on $[0, 2]$. A secant line is the line through the two points

$$(0, f(0)) \quad \text{and} \quad (2, f(2)).$$

Tangent line at $x = -3$. A tangent line is the line that just touches the curve at $x = -3$ and has the same instantaneous slope as the graph there.

Tangent line at $x = 0$. Similarly, the tangent line at $x = 0$ is the line with slope equal to the instantaneous rate of change at $x = 0$.

(b) Average rate of change

Recall that the average rate of change of f on $[a, b]$ is

$$\frac{f(b) - f(a)}{b - a}.$$

On $[-3, -1]$ From the graph, a reasonable estimate is

$$f(-3) \approx 0 \quad \text{and} \quad f(-1) \approx 2.4.$$

Thus

$$\frac{f(-1) - f(-3)}{-1 - (-3)} \approx \frac{2.4 - 0}{2} = 1.2.$$

So the average rate of change on $[-3, -1]$ is approximately

$$\boxed{1.2 \text{ (about 1 to 1.5 is reasonable).}}$$

On $[0, 2]$ From the graph, a reasonable estimate is

$$f(0) \approx 2.0 \quad \text{and} \quad f(2) \approx 1.0.$$

Thus

$$\frac{f(2) - f(0)}{2 - 0} \approx \frac{1.0 - 2.0}{2} = -0.5.$$

So the average rate of change on $[0, 2]$ is approximately

$$\boxed{-0.5 \text{ (about } -0.4 \text{ to } -0.6 \text{ is reasonable).}}$$

Relation to part (a) The average rate of change on an interval is exactly the *slope of the secant line* over that interval. So the numbers above are the slopes of the two secant lines you drew in part (a).

(c) Instantaneous rate of change

The instantaneous rate of change at a point is the slope of the tangent line there.

At $x = -3$ From the graph, the curve is rising steeply at $x = -3$, so the tangent slope is positive and fairly large. A reasonable estimate is

$$\boxed{f'(-3) \approx 3.}$$

At $x = 0$ At $x = 0$, the curve is decreasing but not very steeply. A reasonable estimate is

$$\boxed{f'(0) \approx -0.7 \text{ (values near } -0.5 \text{ to } -1 \text{ are reasonable).}}$$

Relation to part (a) These are the slopes of the tangent lines from part (a). So:

- average rate of change $\longleftrightarrow$ secant slope,
- instantaneous rate of change $\longleftrightarrow$ tangent slope.

Problem 2

We are asked to sketch continuous functions with the given average and instantaneous rates of change. Since the problem only asks for a possible graph, we do not need an explicit formula. We only need to draw a continuous curve that has the required secant slopes and tangent slopes.

(a) A possible graph for f

We want a continuous function f such that:

- the average rate of change on $[-3, 0]$ is -2,
- the average rate of change on $[1, 3]$ is 0.5,
- $f'(-1) = -1$,
- $f'(2) = 1$.

Explanation

The condition

$$\frac{f(0) - f(-3)}{0 - (-3)} = -2$$

means the secant line from $x = -3$ to $x = 0$ must have slope -2. So, for example, we can make the graph pass through the points

$$(-3, 6) \quad \text{and} \quad (0, 0),$$

since

$$\frac{0 - 6}{0 - (-3)} = \frac{-6}{3} = -2.$$

The condition

$$\frac{f(3) - f(1)}{3 - 1} = 0.5$$

means the secant line from $x = 1$ to $x = 3$ must have slope 0.5. So we may choose points such as

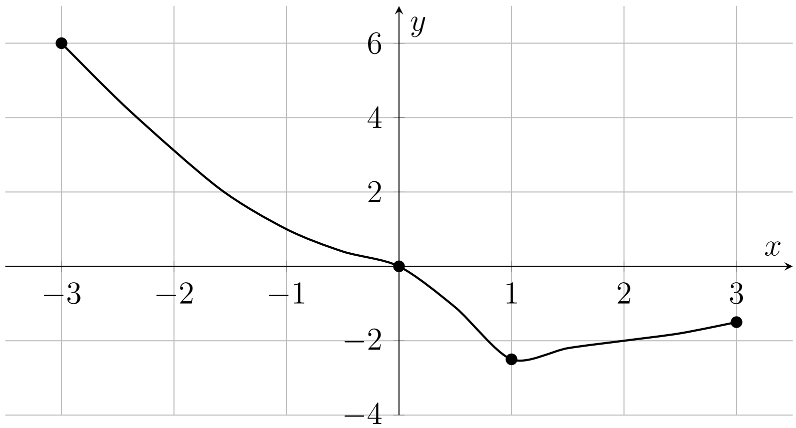
$$(1, -2.5) \quad \text{and} \quad (3, -1.5),$$

because

$$\frac{-1.5 - (-2.5)}{3 - 1} = \frac{1}{2} = 0.5.$$

We also want the tangent slope at $x = -1$ to be -1, and the tangent slope at $x = 2$ to be 1. So we draw the curve so that near $x = -1$ it is decreasing with slope about -1, and near $x = 2$ it is increasing with slope about 1.

Thus the following graph is one valid example.



This graph is continuous, has secant slope -2 from $x = -3$ to $x = 0$, has secant slope 0.5 from $x = 1$ to $x = 3$, and is drawn so that the tangent slopes at $x = -1$ and $x = 2$ match the required values.

(b) A possible graph for g

We want a continuous function g such that:

- $$\frac{g(3) - g(-2)}{5} = 0,$$
  so $g(3) = g(-2)$,
- $$\frac{g(1) - g(-1)}{2} = -1,$$
  so $g(1) - g(-1) = -2$,
- $g'(2) = 1$,
- $g'(-1) = 0$.

Explanation

The condition
$$\frac{g(3) - g(-2)}{5} = 0$$
means the secant line from $x = -2$ to $x = 3$ is horizontal, so the graph should have the same height at those two points. For instance, we can choose

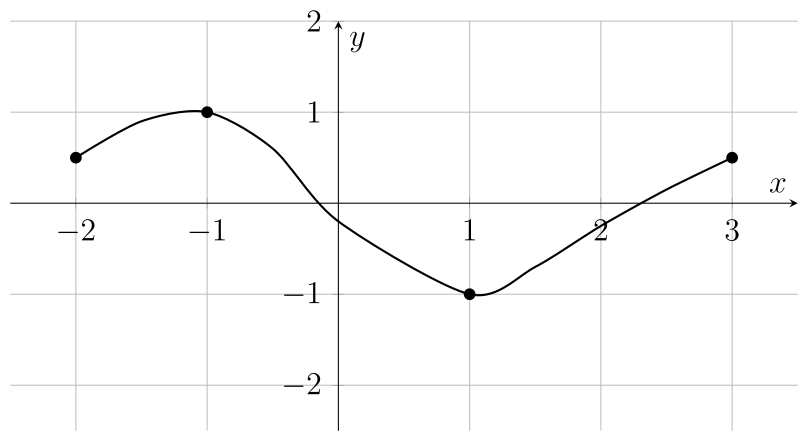
$$(-2, 0.5) \quad \text{and} \quad (3, 0.5).$$

The condition
$$\frac{g(1) - g(-1)}{2} = -1$$
means
$$g(1) - g(-1) = -2.$$
So if we choose
$$g(-1) = 1, \qquad g(1) = -1,$$
then this requirement is satisfied.

The condition $g'(-1) = 0$ means the tangent line is horizontal at $x = -1$, so the graph should flatten there.

The condition $g'(2) = 1$ means the tangent line at $x = 2$ should rise with slope 1.

So the following graph is one possible example.



This graph is continuous, gives equal endpoint values at $x = -2$ and $x = 3$, gives average rate of change -1 from $x = -1$ to $x = 1$, has a horizontal tangent at $x = -1$, and is drawn so that the tangent slope at $x = 2$ is about 1.

Problem 3

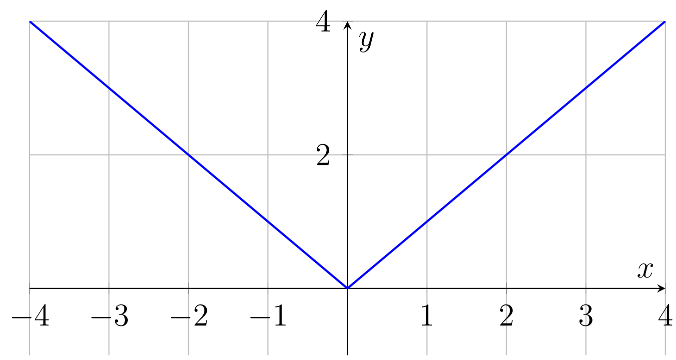
Let

$$h(x) = |x|.$$

(a) Graph the function

A graph of $h(x) = |x|$ is the familiar V-shape:

$$h(x) = \begin{cases} -x, & x < 0, \\ x, & x \geq 0. \end{cases}$$



(b) Why is h differentiable for $x \neq 0$?

For $x > 0$, we have

$$h(x) = x,$$

which is a line, so its derivative exists everywhere and equals 1.

For $x < 0$, we have

$$h(x) = -x,$$

which is also a line, so its derivative exists everywhere and equals -1.

Therefore, h is differentiable at every point except possibly $x = 0$. Visually, the graph is smooth on each side of 0, but it has a sharp corner at 0.

(c) Use the limit definition to show that $h'(0) = \lim_{h \to 0} \frac{|h|}{h}$

By the definition of the derivative at $x = 0$,

$$h'(0) = \lim_{h \to 0} \frac{h(0+h) - h(0)}{h}.$$

Since

$$h(0) = |0| = 0 \quad \text{and} \quad h(0+h) = |h|,$$

we get

$$h'(0) = \lim_{h \to 0} \frac{|h| - 0}{h} = \lim_{h \to 0} \frac{|h|}{h}.$$

So

$$\boxed{h'(0) = \lim_{h \to 0} \frac{|h|}{h}.}$$

(d) Explain why $h'(0)$ fails to exist

Now consider small positive and negative values of h.

If $h > 0$ Then $|h| = h$, so

$$\frac{|h|}{h} = \frac{h}{h} = 1.$$

If $h < 0$ Then $|h| = -h$, so

$$\frac{|h|}{h} = \frac{-h}{h} = -1.$$

Therefore,

$$\lim_{h \to 0^+} \frac{|h|}{h} = 1 \qquad \text{but} \qquad \lim_{h \to 0^-} \frac{|h|}{h} = -1.$$

Since the right-hand and left-hand limits are different, the two-sided limit does not exist. Hence

$$\boxed{h'(0) \text{ does not exist.}}$$

Geometric explanation This agrees with the graph: at $x = 0$, the graph has a corner. The slope coming from the left is -1, while the slope coming from the right is 1, so there is no single tangent slope at $x = 0$.