Math 9A Worksheet 5 Solutions

Problem 1: Deriving $\dfrac{d}{dx}(\sin x) = \cos x$

Recall the derivative definition:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

1(a) Explain each step of the computation

We are given:

$$\begin{aligned}
\frac{d}{dx}[\sin x] &= \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h} && (1)\\
&= \lim_{h \to 0} \frac{\sin x \cos h + \sin h \cos x - \sin x}{h} && (2)\\
&= \lim_{h \to 0} \frac{\sin x(\cos h - 1) + \sin h \cos x}{h} && (3)\\
&= \lim_{h \to 0} \frac{\sin x(\cos h - 1)}{h} + \lim_{h \to 0} \frac{\sin h \cos x}{h} && (4)\\
&= \sin x \lim_{h \to 0} \frac{\cos h - 1}{h} + \cos x \lim_{h \to 0} \frac{\sin h}{h} && (5)\\
&= \sin x \cdot 0 + \cos x \cdot 1. && (6)
\end{aligned}$$

Here is a one-sentence explanation for each step:

- **From (1) to (2):** We use the angle addition identity

$$\sin(x+h) = \sin x \cos h + \cos x \sin h.$$

- **From (2) to (3):** We factor $\sin x$ from the first and third terms:

$$\sin x \cos h - \sin x = \sin x(\cos h - 1).$$

- **From (3) to (4):** We split the fraction into two fractions and use the sum law for limits.
- **From (4) to (5):** Since $\sin x$ and $\cos x$ do not depend on h, they are constants with respect to the limit and can be factored out.

- **From (5) to (6):** We use the standard trigonometric limits

$$\lim_{h\to 0}\frac{\sin h}{h}=1 \qquad \text{and} \qquad \lim_{h\to 0}\frac{\cos h-1}{h}=0.$$

Therefore,

$$\frac{d}{dx}(\sin x)=\cos x.$$

1(b) Use tangent slopes to graph $f'(x)$ when $f(x)=\sin x$

We are asked to estimate the derivative of $\sin x$ using tangent slopes.

Step 1: Slopes at key points

At the key x-values, the tangent slopes of $y=\sin x$ are:

x	-2π	$-\frac{3\pi}{2}$	$-\pi$	$-\frac{\pi}{2}$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$f'(x)$	1	0	-1	0	1	0	-1	0	1

These are exactly the values of $\cos x$ at those points.

Step 2: Check with the limit definition at $x=0$

Using the derivative definition,

$$f'(0)=\lim_{h\to 0}\frac{\sin(0+h)-\sin 0}{h}=\lim_{h\to 0}\frac{\sin h}{h}=1.$$

This agrees with the visual tangent slope at $x=0$.

Since sine is periodic with period 2π, the tangent slope pattern also repeats every 2π. Thus,

$$f'(2\pi)=1 \qquad \text{and} \qquad f'(-2\pi)=1.$$

Step 3: Conclusion

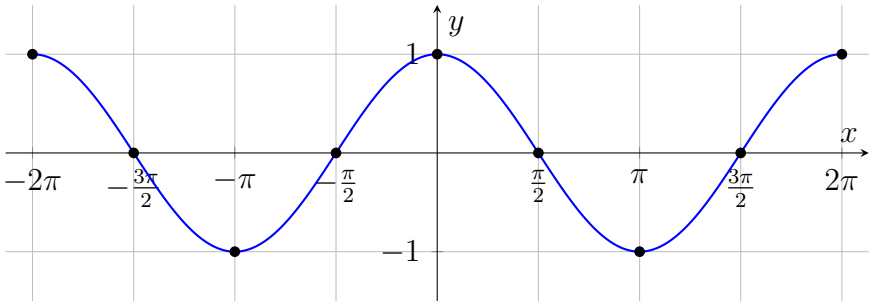
The derivative graph must pass through the points

$$(-2\pi,1),\left(-\frac{3\pi}{2},0\right),(-\pi,-1),\left(-\frac{\pi}{2},0\right),(0,1),\left(\frac{\pi}{2},0\right),(\pi,-1),\left(\frac{3\pi}{2},0\right),(2\pi,1),$$

and the smooth curve through these points is

$$\boxed{f'(x)=\cos x.}$$

Helpful graph



Problem 2: Analyze the graph of $y = p(x)$

From the graph, each piece is a straight line. Reading the picture carefully, the important features are:

- A line segment descends steeply to the point $(-2, 1)$.
- Then another line segment descends to an *open circle* at $(0, 0)$.
- There is a *filled point* at $(0, -2)$.
- Then a rising line segment goes to $(1, -1)$.
- Then a horizontal segment runs from $x = 1$ to $x = 2$ at height -1.
- Then an increasing line segment starts at $(2, -1)$ and has an *open circle* at $x = 3$.

The slopes of the linear pieces are approximately:

$$-3, \quad -\frac{1}{2}, \quad 1, \quad 0, \quad 3.$$

2(a) Values $x = c$ for which $\lim_{x\to c} p(x)$ does not exist

A limit fails to exist when the left-hand and right-hand limits are different.

At $x = 0$, the graph approaches

$$\lim_{x\to 0^-} p(x) = 0 \qquad \text{but} \qquad \lim_{x\to 0^+} p(x) = -2.$$

Since these are different, the two-sided limit does not exist.

At the other key points, the left-hand and right-hand values match.

Therefore,

$$\boxed{\lim_{x\to c} p(x) \text{ does not exist only at } c = 0.}$$

2(b) Values of x for which p is not continuous

A function is not continuous at a point if either the limit does not exist, or the function value does not equal the limit, or the function is not defined there.

- At $x = 0$, the limit does not exist, so p is not continuous.
- At $x = 3$, there is an open circle, so $p(3)$ is not defined; therefore p is not continuous there.

Hence,

$$\boxed{p \text{ is not continuous at } x = 0 \text{ and } x = 3.}$$

2(c) Values of x for which p is not differentiable

A function is not differentiable at:

- corners,
- cusps,
- vertical tangents,
- discontinuities.

From the graph:

- $x = -2$ is a corner where two line segments with different slopes meet.
- $x = 0$ is a discontinuity, so p is not differentiable there.
- $x = 1$ is a corner where the slope changes from 1 to 0.
- $x = 2$ is a corner where the slope changes from 0 to 3.
- $x = 3$ is not in the graph (open circle), so p is not differentiable there.

Thus,

$$\boxed{p \text{ is not differentiable at } x = -2,\ 0,\ 1,\ 2,\ 3.}$$

2(d) Sketch $y = p'(x)$

Since each piece of p is a straight line, the derivative on each interval is just the slope of that line:

$$p'(x) = \begin{cases} -3, & \text{on the leftmost interval } x < -2, \\ -\dfrac{1}{2}, & -2 < x < 0, \\ 1, & 0 < x < 1, \\ 0, & 1 < x < 2, \\ 3, & 2 < x < 3. \end{cases}$$

And $p'(x)$ is undefined at

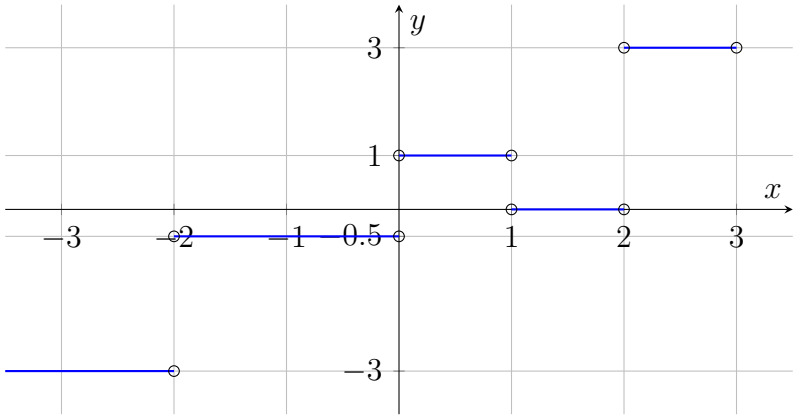
$$x = -2,\ 0,\ 1,\ 2,\ 3.$$

So an accurate graph of $p'(x)$ is a step graph with horizontal pieces at heights

$$-3, \quad -\frac{1}{2}, \quad 1, \quad 0, \quad 3,$$

with open circles at the break points.

Graph of $p'(x)$



Problem 3: Investor and stock value

Let $N(t)$ be the number of shares owned on day t, and let $S(t)$ be the value of one share on day t, measured in dollars per share.

3(a) Formula for total value

The total value is

$$\boxed{V(t) = N(t)S(t).}$$

This makes sense because

$$(\text{shares}) \cdot \left(\frac{\text{dollars}}{\text{share}}\right) = \text{dollars}.$$

3(b) Interpret $N'(t)$ and $S'(t)$

- $N'(t)$ measures the rate at which the number of shares is changing, in units of

$$\text{shares per day.}$$

- $S'(t)$ measures the rate at which the price of one share is changing, in units of

$$\frac{\text{dollars}}{\text{share} \cdot \text{day}}.$$

3(c) Translate the day 100 information into function notation

We are told that on day 100,

- the investor owns 500 shares,
- each share is worth \$31.

So

$$\boxed{N(100) = 500, \qquad S(100) = 31.}$$

3(d) If the investor buys shares at 12 shares/day and price is constant

We are told:

$$N'(100) = 12, \qquad S'(100) = 0.$$

Differentiate

$$V(t) = N(t)S(t)$$

using the product rule:

$$V'(t) = N'(t)S(t) + N(t)S'(t).$$

At $t = 100$,

$$V'(100) = N'(100)S(100) + N(100)S'(100).$$

Substitute:

$$V'(100) = 12(31) + 500(0) = 372.$$

Thus,

$$\boxed{V'(100) = 372 \text{ dollars per day.}}$$

3(e) If the number of shares is constant and price rises by \$0.75/share/day

Now

$$N'(100) = 0, \qquad S'(100) = 0.75.$$

So

$$V'(100) = 0(31) + 500(0.75) = 375.$$

Therefore,

$$\boxed{V'(100) = 375 \text{ dollars per day.}}$$

3(f) If both the number of shares and the price are increasing

Now both effects occur:

$$N'(100) = 12, \qquad S'(100) = 0.75.$$

So

$$V'(100) = 12(31) + 500(0.75) = 372 + 375 = 747.$$

Hence,

$$\boxed{V'(100) = 747 \text{ dollars per day.}}$$

Interpretation

The total increase in value comes from two sources:

- buying more shares, which contributes

$$N'(100)S(100) = 12 \cdot 31 = 372,$$

- the share price rising, which contributes

$$N(100)S'(100) = 500 \cdot 0.75 = 375.$$

Adding them gives the total rate of change:

$$372 + 375 = 747.$$

Problem 4: Sketch derivative graphs from the original graphs

For each graph, we use the idea:

- where f is increasing, $f'(x) > 0$,
- where f is decreasing, $f'(x) < 0$,
- where f has a horizontal tangent, $f'(x) = 0$,
- where f has a corner or cusp, $f'(x)$ does not exist,
- where f is very steep upward, $f'(x)$ is large positive,
- where f is very steep downward, $f'(x)$ is large negative.

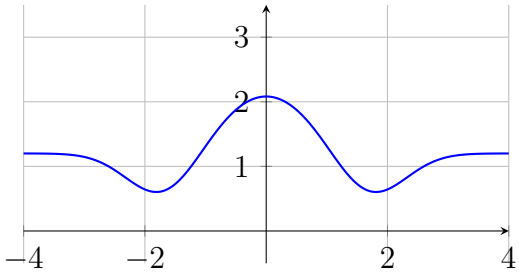
4(a) The graph labeled r

The graph r is an increasing S-shaped curve. It is always increasing, so $r'(x)$ should stay positive. The graph becomes flatter in two places, so the derivative should dip near those places, but remain above the x-axis.

Thus the derivative graph:

- stays above the x-axis,
- has two local minima where the original graph flattens,
- is largest where the original graph is steepest.

A possible sketch is:



4(b) The graph labeled s

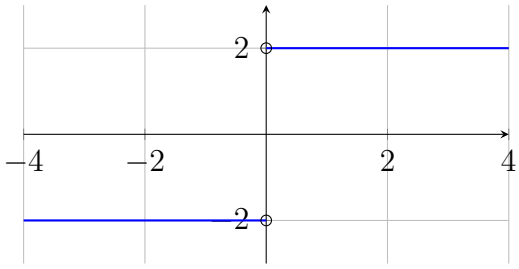
The graph s is a V-shape, like an absolute value graph. The left side is a line with constant negative slope, and the right side is a line with constant positive slope. At the vertex, there is a sharp corner, so the derivative does not exist there.

Thus:

$$s'(x) = \begin{cases} m_1 < 0, & x < 0, \\ m_2 > 0, & x > 0, \end{cases}$$

and $s'(0)$ does not exist.

A possible sketch is:



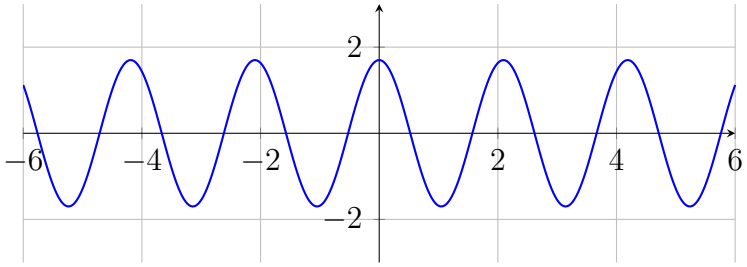
4(c) The graph labeled w

The graph w looks like repeating arches above the x-axis, with sharp points where the curve touches the x-axis. On each arch:

- the graph rises, so the derivative is positive,
- at the top of the arch, the derivative is 0,
- then the graph falls, so the derivative is negative,
- at the sharp bottom points, the derivative does not exist.

So the derivative should oscillate above and below the axis, crossing 0 at the top of each arch, and be undefined at each cusp.

A schematic sketch is:



This is only a schematic shape: the main point is the alternating positive and negative slopes, with breaks at the cusps of w.

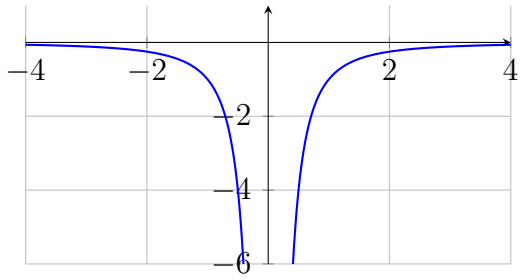
4(d) The graph labeled z

The graph z looks like $1/x$: it has a vertical asymptote at $x = 0$, a positive branch for $x > 0$, and a negative branch for $x < 0$. It is decreasing on both sides of the asymptote, so its derivative is negative on both sides.

Also, near $x = 0$, the graph becomes very steep, so $z'(x)$ becomes very negative. Far from 0, the graph flattens, so $z'(x) \to 0$.

Thus the derivative should look like $-1/x^2$: always negative, symmetric about the y-axis, with a vertical asymptote at $x = 0$.

A possible sketch is:



Overall process for sketching a derivative graph

When sketching the derivative from the graph of the original function, I usually identify these features first:

- where the original graph is increasing or decreasing,
- where it has horizontal tangents,
- where it has corners, cusps, or discontinuities,
- where it is steep and where it is flat.

Then I translate those observations into derivative information:

- increasing $\Rightarrow$ derivative positive,
- decreasing $\Rightarrow$ derivative negative,
- horizontal tangent $\Rightarrow$ derivative zero,
- corner or cusp $\Rightarrow$ derivative undefined,
- steeper graph $\Rightarrow$ larger magnitude of the derivative.

That gives a good qualitative sketch of the derivative, even when no explicit formula is given.