Math 9A Worksheet 6 Solutions

Problem 1

We are given the following table:

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
-1	2	-5	-3	4
2	-3	4	-1	2

We use derivative rules: product rule, quotient rule, chain rule, and basic derivative rules.

(a) If $P(x) = f(x)g(x)$, find $P'(2)$

Use the product rule:

$$P'(x) = f'(x)g(x) + f(x)g'(x).$$

At $x = 2$,

$$P'(2) = f'(2)g(2) + f(2)g'(2).$$

From the table:

$$f'(2) = 4, \quad g(2) = -1, \quad f(2) = -3, \quad g'(2) = 2.$$

So

$$P'(2) = 4(-1) + (-3)(2) = -4 - 6 = -10.$$

$$\boxed{P'(2) = -10}$$

(b) If $Q(x) = \dfrac{f(x)}{g(x)}$, find $Q'(-1)$

Use the quotient rule:

$$Q'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}.$$

At $x = -1$,

$$Q'(-1) = \frac{f'(-1)g(-1) - f(-1)g'(-1)}{(g(-1))^2}.$$

From the table:

$$f'(-1) = -5, \quad g(-1) = -3, \quad f(-1) = 2, \quad g'(-1) = 4.$$

Thus

$$Q'(-1) = \frac{(-5)(-3) - 2(4)}{(-3)^2} = \frac{15 - 8}{9} = \frac{7}{9}.$$

$$\boxed{Q'(-1) = \frac{7}{9}}$$

(c) If $B(x) = x^2 + 7f(x) + 2$, find $B'(-1)$

Differentiate term by term:

$$B'(x) = 2x + 7f'(x).$$

At $x = -1$,

$$B'(-1) = 2(-1) + 7f'(-1).$$

Since $f'(-1) = -5$,

$$B'(-1) = -2 + 7(-5) = -2 - 35 = -37.$$

$$\boxed{B'(-1) = -37}$$

(d) If $C(x) = f(g(x))$, find $C'(2)$

Use the chain rule:

$$C'(x) = f'(g(x))\, g'(x).$$

At $x = 2$,

$$C'(2) = f'(g(2))\, g'(2).$$

From the table:

$$g(2) = -1, \qquad g'(2) = 2.$$

So

$$C'(2) = f'(-1) \cdot 2.$$

Since $f'(-1) = -5$,

$$C'(2) = -5 \cdot 2 = -10.$$

$$\boxed{C'(2) = -10}$$

(e) If $D(x) = f(f(x))$, find $D'(-1)$

Again use the chain rule:

$$D'(x) = f'(f(x))\, f'(x).$$

At $x = -1$,

$$D'(-1) = f'(f(-1))\, f'(-1).$$

From the table:

$$f(-1) = 2, \qquad f'(-1) = -5.$$

So

$$D'(-1) = f'(2) \cdot (-5).$$

Since $f'(2) = 4$,

$$D'(-1) = 4(-5) = -20.$$

$$\boxed{D'(-1) = -20}$$

(f) If $E(x) = f(g(e^x))$, find $E'(0)$

This is a composition of three functions, so we apply the chain rule carefully:

$$E'(x) = f'(g(e^x)) \cdot g'(e^x) \cdot e^x.$$

At $x = 0$,

$$e^0 = 1.$$

So

$$E'(0) = f'(g(1)) \cdot g'(1) \cdot 1.$$

But the table only gives information about f and g at

$$x = -1 \quad \text{and} \quad x = 2.$$

We do *not* know $g(1)$ or $g'(1)$, so there is not enough information to evaluate $E'(0)$.

$$\boxed{\text{Not enough information}}$$

Problem 2

The worksheet gives graphs of f and g. We will use only the information that can be read directly from the graphs: point values, slopes of line segments, and where corners occur. We do **not** need explicit formulas for f and g.

From the graph, we observe the following:

- For f:

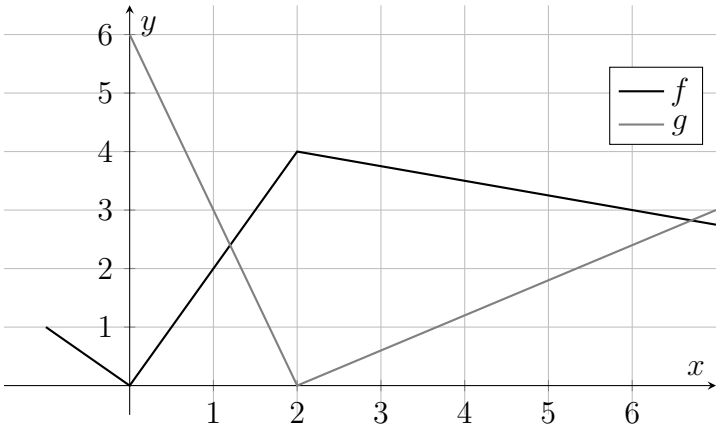
- on the segment from $x = 0$ to $x = 2$, the graph goes through $(0, 0)$ and $(2, 4)$, so its slope is
$$\frac{4-0}{2-0} = 2.$$
- for $x > 2$, the graph is a decreasing line segment, and from the graph its slope is
$$-\frac{1}{4}.$$
- there is a corner at $x = 2$, so $f'(2)$ does not exist.

- For g:
  - on the segment from $x = 0$ to $x = 2$, the graph goes through $(0, 6)$ and $(2, 0)$, so its slope is
$$\frac{0-6}{2-0} = -3.$$
  - for $x > 2$, the graph is an increasing line segment, and from the graph its slope is
$$\frac{3}{5}.$$
  - there is a corner at $x = 2$, so $g'(2)$ does not exist.

We will repeatedly use the following values read from the graph:

$$f(1) = 2, \qquad f(2) = 4, \qquad f(4) = \frac{7}{2},$$
$$g(1) = 3, \qquad g(2) = 0, \qquad g(4) = \frac{6}{5}, \qquad g(5) = \frac{9}{5}.$$

Helpful graph



(a) If $q(x) = \dfrac{f(x)}{g(x)}$, find $q'(1)$

Use the quotient rule:

$$q'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}.$$

From the graph at $x = 1$:

$$f(1) = 2, \qquad g(1) = 3.$$

Also, since $x = 1$ lies on the line segment of f from $(0, 0)$ to $(2, 4)$, its slope is

$$f'(1) = 2.$$

Since $x = 1$ lies on the line segment of g from $(0, 6)$ to $(2, 0)$, its slope is

$$g'(1) = -3.$$

Now substitute:

$$q'(1) = \frac{(2)(3) - (2)(-3)}{3^2} = \frac{6 + 6}{9} = \frac{12}{9} = \frac{4}{3}.$$

$$\boxed{q'(1) = \frac{4}{3}}$$

(b) If $p(x) = f(x)g(x)$, find $p'(2)$

At $x = 2$, both graphs have corners, so $f'(2)$ and $g'(2)$ do not exist. That means we must check the left-hand and right-hand derivatives of the product.

$$\boxed{p'(2) \text{ does not exist}}$$

(c) If $u(x) = f(g(x))$, find $u'(4)$

Use the chain rule:

$$u'(x) = f'(g(x))\, g'(x).$$

From the graph, when $x = 4$, the point on g is

$$g(4) = \frac{6}{5}.$$

Since $\frac{6}{5}$ lies between 0 and 2, the input $g(4)$ lands on the middle line segment of f, whose slope is

$$f'\left(\frac{6}{5}\right) = 2.$$

Also, for $x > 2$, the graph of g is a line segment of slope

$$g'(4) = \frac{3}{5}.$$

So

$$u'(4) = f'(g(4))\, g'(4) = 2 \cdot \frac{3}{5} = \frac{6}{5}.$$

$$\boxed{u'(4) = \frac{6}{5}}$$

(d) If $u(x) = f(g(x))$, find $u'(5)$

Again,

$$u'(x) = f'(g(x))\, g'(x).$$

From the graph,

$$g(5) = \frac{9}{5}.$$

Since $\frac{9}{5}$ is still between 0 and 2, this input also lands on the same middle line segment of f. Therefore

$$f'\left(\frac{9}{5}\right) = 2.$$

Also,

$$g'(5) = \frac{3}{5}.$$

Thus

$$u'(5) = 2 \cdot \frac{3}{5} = \frac{6}{5}.$$

$$\boxed{u'(5) = \frac{6}{5}}$$

(e) If $v(x) = g(f(x))$, find $v'(4)$

Use the chain rule:

$$v'(x) = g'(f(x))\, f'(x).$$

From the graph,

$$f(4) = \frac{7}{2}.$$

Since $\frac{7}{2} > 2$, this input lands on the right-hand line segment of g, whose slope is

$$g'\left(\frac{7}{2}\right) = \frac{3}{5}.$$

Also, for $x > 2$, the graph of f has slope

$$f'(4) = -\frac{1}{4}.$$

Therefore,

$$v'(4) = g'(f(4))\, f'(4) = \frac{3}{5}\left(-\frac{1}{4}\right) = -\frac{3}{20}.$$

$$\boxed{v'(4) = -\frac{3}{20}}$$

(f) If $w(x) = g(g(x))$, find $w'(4)$

Use the chain rule:

$$w'(x) = g'(g(x))\, g'(x).$$

From the graph,

$$g(4) = \frac{6}{5}.$$

Since $\frac{6}{5} < 2$, this input lands on the left line segment of g, whose slope is

$$g'\left(\frac{6}{5}\right) = -3.$$

Also,

$$g'(4) = \frac{3}{5}.$$

Hence

$$w'(4) = (-3)\left(\frac{3}{5}\right) = -\frac{9}{5}.$$

$$\boxed{w'(4) = -\frac{9}{5}}$$

Problem 3

The height of the boat above the sea floor is

$$y = 33 + \sin(\pi t),$$

where t is measured in minutes.

(a) Graph the function

This is a sine wave shifted upward by 33 units.

- Midline: $y = 33$
- Amplitude: 1
- Period:

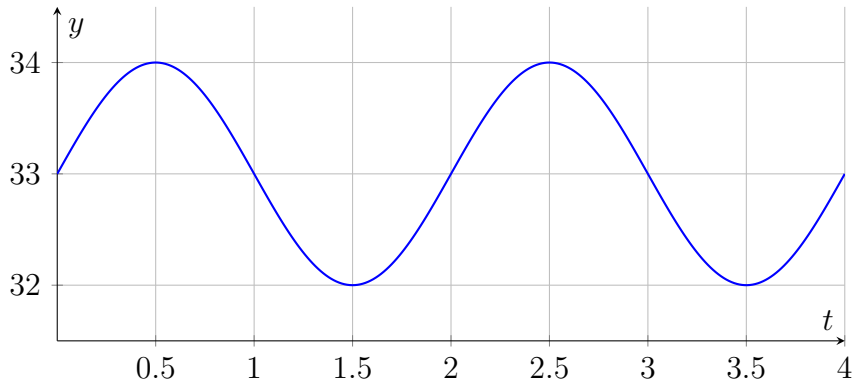
$$\frac{2\pi}{\pi} = 2$$

So the graph oscillates between

$$32 \quad \text{and} \quad 34$$

and repeats every 2 minutes.

Graph



(b) At $t = 0$, is the boat moving up or down?

To decide whether the boat is moving up or down, we look at the derivative.

Differentiate:

$$y'(t) = \pi \cos(\pi t).$$

At $t = 0$,

$$y'(0) = \pi \cos(0) = \pi > 0.$$

A positive derivative means the height is increasing, so the boat is moving upward.

$$\boxed{\text{At } t = 0, \text{ the boat is moving up.}}$$

(c) At $t = 1$, is the boat moving up or down?

Evaluate the derivative:

$$y'(1) = \pi \cos(\pi) = \pi(-1) = -\pi < 0.$$

A negative derivative means the height is decreasing, so the boat is moving downward.

$$\boxed{\text{At } t = 1, \text{ the boat is moving down.}}$$

(d) At $t = 2$, is the boat moving up or down?

Evaluate:

$$y'(2) = \pi \cos(2\pi) = \pi > 0.$$

Again the derivative is positive, so the boat is moving upward.

$$\boxed{\text{At } t = 2, \text{ the boat is moving up.}}$$

(e) Find the vertical velocity v of the boat at time t

Velocity is the derivative of position:

$$v(t) = y'(t).$$

Since

$$y = 33 + \sin(\pi t),$$

we get

$$v(t) = \pi \cos(\pi t).$$

$$\boxed{v(t) = \pi \cos(\pi t)}$$

(f) Find $v'(0)$. How does this relate to your answer in (b)?

Differentiate velocity:

$$v'(t) = \frac{d}{dt}\big(\pi \cos(\pi t)\big) = -\pi^2 \sin(\pi t).$$

At $t = 0$,

$$v'(0) = -\pi^2 \sin(0) = 0.$$

So

$$\boxed{v'(0) = 0}$$

This means the velocity is momentarily not changing at $t = 0$. In part (b), we found the boat is moving up because

$$v(0) = \pi > 0.$$

So the boat is moving upward, and at that instant its upward velocity is at a local maximum.

(g) Find $v'(1)$. How does this relate to your answer in (c)?

Using

$$v'(t) = -\pi^2 \sin(\pi t),$$

we get

$$v'(1) = -\pi^2 \sin(\pi) = 0.$$

Thus

$$\boxed{v'(1) = 0}$$

This matches part (c): the boat is moving downward because

$$v(1) = -\pi < 0,$$

and at that instant the downward velocity is momentarily not changing. In fact, it is at a local minimum value.