Math 9A Worksheet 8 Solutions

Problem 1

Let

$$h(x) = \frac{x^5 + x - 2}{x^2 - 1}.$$

We will study the limit as $x \to 1$ and connect it to the idea behind L'Hospital's Rule.

(a) What is the domain of h?

The denominator cannot be zero, so we require

$$x^2 - 1 \neq 0.$$

Factor:

$$x^2 - 1 = (x - 1)(x + 1).$$

Thus the denominator is zero at

$$x = 1 \quad \text{and} \quad x = -1.$$

Therefore, the domain is

$$\boxed{(-\infty, -1) \cup (-1, 1) \cup (1, \infty).}$$

(b) Explain why $\lim_{x \to 1} \frac{x^5 + x - 2}{x^2 - 1}$ is indeterminate

Substitute $x = 1$ into the numerator and denominator:

$$1^5 + 1 - 2 = 0, \qquad 1^2 - 1 = 0.$$

So the limit takes the form

$$\frac{0}{0},$$

which is an indeterminate form.

Thus

$$\boxed{\lim_{x \to 1} \frac{x^5 + x - 2}{x^2 - 1} \text{ is of type } \frac{0}{0}.}$$

(c) Find the tangent lines L_f and L_g at $x = 1$

Let

$$f(x) = x^5 + x - 2, \qquad g(x) = x^2 - 1.$$

We first compute the values at $x = 1$:

$$f(1) = 1 + 1 - 2 = 0, \qquad g(1) = 1 - 1 = 0.$$

Now compute the derivatives:

$$f'(x) = 5x^4 + 1, \qquad g'(x) = 2x.$$

So at $x = 1$,

$$f'(1) = 5(1)^4 + 1 = 6, \qquad g'(1) = 2(1) = 2.$$

The tangent line to f at $x = 1$ is

$$L_f(x) = f(1) + f'(1)(x - 1) = 0 + 6(x - 1) = 6x - 6.$$

The tangent line to g at $x = 1$ is

$$L_g(x) = g(1) + g'(1)(x - 1) = 0 + 2(x - 1) = 2x - 2.$$

Therefore,

$$\boxed{L_f(x) = 6x - 6, \qquad L_g(x) = 2x - 2.}$$

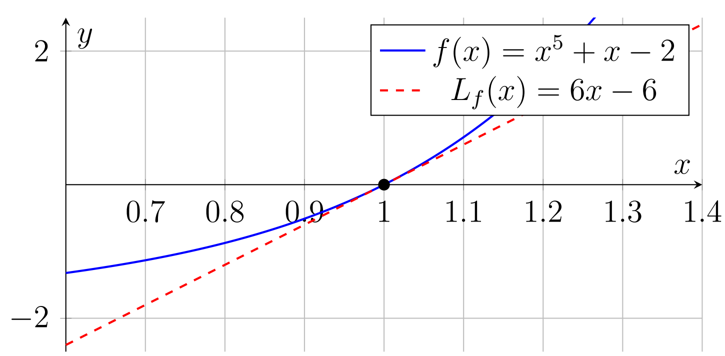
(d) What do you notice when graphing f and L_f near $x = 1$?

Near $x = 1$, the graph of $f(x)$ and its tangent line $L_f(x)$ become almost indistinguishable. This is the basic idea of linearization: for inputs close to the point of tangency, a differentiable function is well-approximated by its tangent line.

In other words,

$$f(x) \approx L_f(x) \quad \text{when } x \text{ is near } 1.$$

Illustration



(e) Explain why $h(x) \approx \dfrac{L_f(x)}{L_g(x)}$ near $x = 1$

Since both f and g are differentiable at $x = 1$, each is well-approximated by its tangent line near that point:

$$f(x) \approx L_f(x), \qquad g(x) \approx L_g(x).$$

So for x near 1,

$$h(x) = \frac{f(x)}{g(x)} \approx \frac{L_f(x)}{L_g(x)}.$$

This approximation is especially useful here because both $f(1)$ and $g(1)$ are zero, so their linear terms control the local behavior of the ratio.

(f) Compute $\lim_{x \to 1} \dfrac{L_f(x)}{L_g(x)}$

Using the tangent lines,

$$\frac{L_f(x)}{L_g(x)} = \frac{6x - 6}{2x - 2}.$$

Factor:

$$\frac{6x - 6}{2x - 2} = \frac{6(x-1)}{2(x-1)} = 3 \quad \text{for } x \neq 1.$$

Therefore,

$$\lim_{x \to 1} \frac{L_f(x)}{L_g(x)} = 3.$$

So we expect

$$\boxed{\lim_{x \to 1} h(x) = 3.}$$

(g) Investigate $h(x)$ graphically near $x = 1$

We can also confirm this algebraically by factoring the numerator:

$$x^5 + x - 2 = (x - 1)(x^4 + x^3 + x^2 + x + 2),$$

and

$$x^2 - 1 = (x - 1)(x + 1).$$

So for $x \neq 1$,

$$h(x) = \frac{x^4 + x^3 + x^2 + x + 2}{x + 1}.$$

Now substitute $x = 1$:

$$\lim_{x \to 1} h(x) = \frac{1 + 1 + 1 + 1 + 2}{2} = \frac{6}{2} = 3.$$

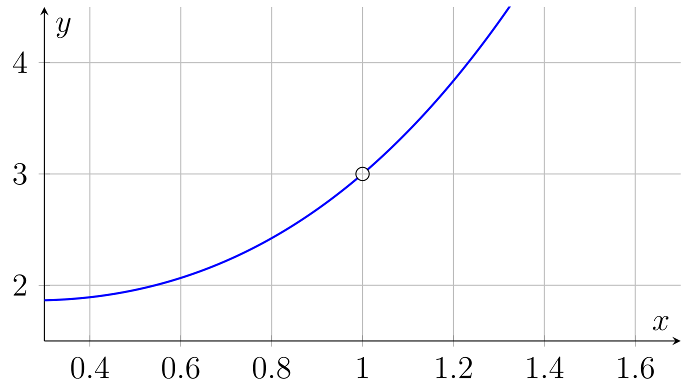
Thus,

$$\boxed{\lim_{x \to 1} h(x) = 3.}$$

The graph has a *hole* at $x = 1$, because the original formula is undefined there, but the nearby values approach 3. So the graph has a removable discontinuity at the point

$$\boxed{(1, 3).}$$

Graph of h near $x = 1$



Problem 2

In this problem, we reason graphically from the figures.

(a) Use the graph on the left

From the graph, both curves pass through the same point at $x = 2$, namely

$$f(2) = 0, \qquad g(2) = 0.$$

The tangent line drawn for f at $x = 2$ has a mild positive slope. From the grid, its rise is about 1 when the run is about 2, so

$$f'(2) \approx \frac{1}{2}.$$

The tangent line drawn for g at $x = 2$ is much steeper. From the grid, its rise is about 4 for a run of 1, so

$$g'(2) \approx 4.$$

Thus

$$f(2) = 0, \qquad f'(2) \approx \frac{1}{2}, \qquad g(2) = 0, \qquad g'(2) \approx 4.$$

Because both function values are zero, the quotient is a $0/0$ form. Graphically, near $x = 2$, the two curves behave like their tangent lines, so

$$\frac{f(x)}{g(x)} \approx \frac{f'(2)(x-2)}{g'(2)(x-2)} = \frac{f'(2)}{g'(2)}.$$

Therefore,

$$\lim_{x \to 2} \frac{f(x)}{g(x)} \approx \frac{\frac{1}{2}}{4} = \frac{1}{8}.$$

So the graph suggests

$$\boxed{\lim_{x \to 2} \frac{f(x)}{g(x)} \approx \frac{1}{8}.}$$

(b) Use the graph on the right

From the graph, both curves meet at the same point when $x = 2$. Reading the grid,

$$p(2) = \frac{3}{2}, \qquad q(2) = \frac{3}{2}.$$

The graph of q has a horizontal tangent there, so

$$q'(2) = 0.$$

The graph of p has a negative tangent slope there; from the grid it appears about

$$p'(2) \approx -1.$$

Thus

$$p(2) = \frac{3}{2}, \qquad p'(2) \approx -1, \qquad q(2) = \frac{3}{2}, \qquad q'(2) = 0.$$

To find the limit,

$$\lim_{x \to 2} \frac{p(x)}{q(x)},$$

we note that $q(2) = \frac{3}{2} \neq 0$. Since the denominator is not zero, we can use direct substitution:

$$\lim_{x \to 2} \frac{p(x)}{q(x)} = \frac{p(2)}{q(2)} = \frac{\frac{3}{2}}{\frac{3}{2}} = 1.$$

Therefore,

$$\boxed{\lim_{x \to 2} \frac{p(x)}{q(x)} = 1.}$$

Graph comments

This problem shows two different situations:

- In part (a), both numerator and denominator go to 0, so the ratio is controlled by the tangent slopes.
- In part (b), the denominator does *not* go to 0, so the quotient limit is just the quotient of the function values.

Problem 3

We are asked to determine the nature of

$$\lim_{x \to a} \frac{f(x)}{g(x)},$$

where $g(x)$ is the blue line and $f(x)$ is the black curve.

From the picture:

- the black curve touches the x-axis at $x = a$, so $f(a) = 0$,
- the blue line crosses the x-axis at $x = a$, so $g(a) = 0$,
- near $x = a$, the black curve behaves like a parabola opening upward, so $f(x) \geq 0$ on both sides of a,
- near $x = a$, the blue line changes sign because it crosses the axis.

So near $x = a$,

$$f(x) \sim C(x-a)^2 \quad \text{and} \quad g(x) \sim m(x-a),$$

where $C > 0$ and $m \neq 0$.

Therefore,

$$\frac{f(x)}{g(x)} \sim \frac{C(x-a)^2}{m(x-a)} = \frac{C}{m}(x-a).$$

As $x \to a$, this tends to 0.

So the limit is

$$\boxed{0.}$$

Why it is not undefined

Although the quotient changes sign from one side to the other, both sides still approach 0. So the two-sided limit exists and equals 0.

Problem 4

Below, the graphs of f and g near $x = a$ are shown together with their tangent line approximations L_f and L_g at $x = a$.

We explain why each line is correct:

$$\begin{aligned}
\lim_{x \to a} \frac{f(x)}{g(x)} &= \lim_{x \to a} \frac{L_f(x)}{L_g(x)} && (1)\\
&= \lim_{x \to a} \frac{f'(a)(x-a) + f(a)}{g'(a)(x-a) + g(a)} && (2)\\
&= \lim_{x \to a} \frac{f'(a)(x-a)}{g'(a)(x-a)} && (3)\\
&= \lim_{x \to a} \frac{f'(a)}{g'(a)}. && (4)
\end{aligned}$$

Why line (1) is correct

Near $x = a$, differentiable functions are well approximated by their tangent lines. So

$$f(x) \approx L_f(x), \qquad g(x) \approx L_g(x).$$

That means the quotient

$$\frac{f(x)}{g(x)}$$

has essentially the same limiting behavior as

$$\frac{L_f(x)}{L_g(x)}.$$

This is the geometric idea behind L'Hospital's Rule.

Why line (2) is correct

The tangent line to f at $x = a$ is

$$L_f(x) = f(a) + f'(a)(x-a),$$

and the tangent line to g at $x = a$ is

$$L_g(x) = g(a) + g'(a)(x-a).$$

Substituting these into line (1) gives

$$\frac{L_f(x)}{L_g(x)} = \frac{f'(a)(x-a) + f(a)}{g'(a)(x-a) + g(a)}.$$

Why line (3) is correct

In the pictured situation, both graphs pass through the x-axis at $x = a$. So

$$f(a) = 0, \qquad g(a) = 0.$$

Substituting this into line (2) gives

$$\frac{f'(a)(x-a)+0}{g'(a)(x-a)+0} = \frac{f'(a)(x-a)}{g'(a)(x-a)}.$$

Why line (4) is correct

For $x \neq a$, we can cancel the factor $x - a$:

$$\frac{f'(a)(x-a)}{g'(a)(x-a)} = \frac{f'(a)}{g'(a)}.$$

Since the expression is constant for all nearby $x \neq a$, the limit is simply

$$\frac{f'(a)}{g'(a)}.$$

Thus

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}.$$

In words: why L'Hospital's Rule is true

When two differentiable functions both approach 0 at the same point, their values near that point are mainly determined by their tangent lines. The tangent lines are controlled by the derivatives, so the ratio of the functions behaves like the ratio of their slopes:

$$\frac{f(x)}{g(x)} \approx \frac{f'(a)(x-a)}{g'(a)(x-a)} = \frac{f'(a)}{g'(a)}.$$

So L'Hospital's Rule works because, near the point of interest, differentiable functions look linear, and the linear terms dominate the quotient.

$$\boxed{\text{Near } x = a, \text{ the quotient of the functions behaves like the quotient of their tangent lines.}}$$