Math 009B
Week 10 Worksheet Solutions by Khoi Vo

Problem 1

Find the area enclosed by

$$f(x) = 2x^2 + 5x - 3, \qquad g(x) = x^2 + 4x - 1.$$

First find the intersection points.

$$\begin{aligned} 2x^2 + 5x - 3 &= x^2 + 4x - 1 \\ x^2 + x - 2 &= 0 \\ (x+2)(x-1) &= 0 \end{aligned}$$

Thus the curves intersect at

$$x = -2, \quad x = 1.$$

Now determine which function is on top.

$$\begin{aligned} f(x) - g(x) &= (2x^2 + 5x - 3) - (x^2 + 4x - 1) \\ &= x^2 + x - 2 \end{aligned}$$

Testing $x = 0$:

$$f(0) - g(0) = -2 < 0$$

so $g(x)$ is above $f(x)$ on $[-2, 1]$.
Therefore the enclosed area is

$$A = \int_{-2}^{1} \big(g(x) - f(x)\big)\, dx$$

$$\begin{aligned} A &= \int_{-2}^{1} [(x^2 + 4x - 1) - (2x^2 + 5x - 3)] dx \\ &= \int_{-2}^{1} (-x^2 - x + 2)\, dx \end{aligned}$$

Integrate:

$$\int(-x^2 - x + 2)dx = -\frac{x^3}{3} - \frac{x^2}{2} + 2x$$

$$A = \left[-\frac{x^3}{3} - \frac{x^2}{2} + 2x\right]_{-2}^{1}$$

Evaluate:

$$F(1) = -\frac{1}{3} - \frac{1}{2} + 2 = \frac{7}{6}$$

$$F(-2) = \frac{8}{3} - 2 - 4 = -\frac{10}{3}$$

Thus

$$\begin{aligned} A &= \frac{7}{6} - \left(-\frac{10}{3}\right) \\ &= \frac{7}{6} + \frac{20}{6} \\ &= \frac{27}{6} \\ &= \frac{9}{2} \end{aligned}$$

$$\boxed{A = \frac{9}{2}}$$

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Problem 2

Find the area enclosed by

$$f(x) = x, \qquad g(x) = \sqrt{x}.$$

First find the intersection points.

$$\begin{aligned} x &= \sqrt{x} \\ x^2 &= x \\ x(x-1) &= 0 \end{aligned}$$

So

$$x = 0, \quad x = 1$$

On the interval $[0, 1]$,

$$\sqrt{x} \geq x$$

Thus the area is

$$A = \int_0^1 (\sqrt{x} - x)\, dx$$

Write $\sqrt{x} = x^{1/2}$:

$$A = \int_0^1 (x^{1/2} - x)\, dx$$

Integrate:

$$\int x^{1/2} dx = \frac{2}{3} x^{3/2}, \qquad \int x\, dx = \frac{x^2}{2}$$

$$A = \left[\frac{2}{3} x^{3/2} - \frac{x^2}{2} \right]_0^1$$

$$A = \frac{2}{3} - \frac{1}{2}$$

$$A = \frac{4}{6} - \frac{3}{6}$$

$$\boxed{A = \frac{1}{6}}$$

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Problem 3

Set up (do not evaluate) the arc length of

$$f(x) = \sqrt{x}$$

on the interval $[0, 1]$.
The arc length formula is

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$$

Compute the derivative.

$$f(x) = x^{1/2}$$

$$f'(x) = \frac{1}{2\sqrt{x}}$$

Substitute into the arc length formula.

$$\begin{aligned} L &= \int_0^1 \sqrt{1 + \left(\frac{1}{2\sqrt{x}}\right)^2}\, dx \\ &= \int_0^1 \sqrt{1 + \frac{1}{4x}}\, dx \end{aligned}$$

$$\boxed{L = \int_0^1 \sqrt{1 + \frac{1}{4x}}\, dx}$$

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Problem 4

Find the surface area formed by revolving

$$y = x^2$$

on the interval $[0, 1]$ about the y-axis.
The surface area formula is

$$S = \int_a^b 2\pi x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Compute the derivative.

$$y = x^2$$

$$\frac{dy}{dx} = 2x$$

Substitute:

$$S = \int_0^1 2\pi x \sqrt{1 + (2x)^2}\, dx$$

$$S = \int_0^1 2\pi x \sqrt{1 + 4x^2}\, dx$$

Let

$$u = 1 + 4x^2$$

$$du = 8x\,dx$$

$$x\,dx = \frac{du}{8}$$

Change the limits.

$$x = 0 \Rightarrow u = 1$$

$$x = 1 \Rightarrow u = 5$$

Thus

$$\begin{aligned} S &= 2\pi \int_0^1 x\sqrt{1+4x^2}\,dx \\ &= \frac{\pi}{4}\int_1^5 u^{1/2}\,du \end{aligned}$$

Integrate:

$$\int u^{1/2} du = \frac{2}{3}u^{3/2}$$

$$S = \frac{\pi}{4}\left[\frac{2}{3}u^{3/2}\right]_1^5$$

$$S = \frac{\pi}{6}(5^{3/2} - 1)$$

Since

$$5^{3/2} = 5\sqrt{5}$$

$$\boxed{S = \frac{\pi}{6}(5\sqrt{5} - 1)}$$