Math 9A Sample Final — Problems 10, 11, 12

Problem 10

Given

$$f'(x) = e^{-2x}(3 - x)(x + 1)^2,$$

find all critical numbers of f, and decide whether each gives a relative maximum, relative minimum, or neither.

Step 1: Find the critical numbers

Critical numbers occur where

$$f'(x) = 0$$

or where $f'(x)$ is undefined.

We factor the derivative:

$$f'(x) = e^{-2x}(3 - x)(x + 1)^2.$$

Now examine each factor carefully:

- $e^{-2x} > 0$ for every real x, so this factor is never zero and never undefined.
- $3 - x = 0$ when $x = 3$.
- $(x + 1)^2 = 0$ when $x = -1$.

Since $f'(x)$ is defined for all real x, the only critical numbers come from solving

$$f'(x) = 0.$$

Therefore the critical numbers are

$$\boxed{x = -1 \quad \text{and} \quad x = 3.}$$

Step 2: Determine the sign of $f'(x)$

To classify the critical numbers, we study the sign of $f'(x)$.

Notice:

- e^{-2x} is always positive,
- $(x+1)^2 \geq 0$ and is positive except at $x = -1$,
- so away from $x = -1$, the sign of $f'(x)$ is controlled by $3 - x$.

Now analyze intervals determined by the critical numbers:

$$(-\infty, -1), \qquad (-1, 3), \qquad (3, \infty).$$

On $(-\infty, -1)$: If $x < -1$, then $3 - x > 0$, so

$$f'(x) > 0.$$

At $x = -1$:

$$f'(-1) = 0.$$

On $(-1, 3)$: If $-1 < x < 3$, then $3 - x > 0$, so

$$f'(x) > 0.$$

At $x = 3$:

$$f'(3) = 0.$$

On $(3, \infty)$: If $x > 3$, then $3 - x < 0$, so

$$f'(x) < 0.$$

Sign chart

x	$(-\infty, -1)$	-1	$(-1, 3)$	3	$(3, \infty)$
e^{-2x}	$+$	$+$	$+$	$+$	$+$
$3 - x$	$+$	$+$	$+$	0	$-$
$(x+1)^2$	$+$	0	$+$	$+$	$+$
$f'(x)$	$+$	0	$+$	0	$-$

Step 3: Classify each critical number

At $x = -1$: The derivative is positive on both sides:

$$f'(x) > 0 \text{ for } x < -1, \qquad f'(x) > 0 \text{ for } x > -1 \text{ (near } -1).$$

So f is increasing before and after $x = -1$.

Therefore f does not change from increasing to decreasing or from decreasing to increasing there.

Hence,

$$\boxed{x = -1 \text{ is neither a relative maximum nor a relative minimum.}}$$

At $x = 3$: The derivative changes from positive to negative:

$$f'(x) > 0 \text{ for } x < 3, \qquad f'(x) < 0 \text{ for } x > 3.$$

So f changes from increasing to decreasing, which means f has a relative maximum at $x = 3$.

Thus,

$$\boxed{x = 3 \text{ is a relative maximum.}}$$

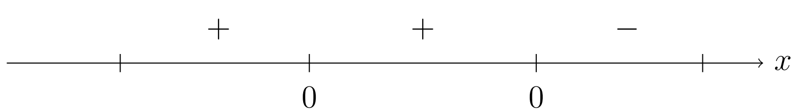
There is no point where the derivative changes from negative to positive, so there is no relative minimum.

Final answer

$$\boxed{\text{Critical numbers: } x = -1,\ x = 3}$$

$$\boxed{x = -1 \text{ is neither}} \qquad \boxed{x = 3 \text{ is a relative maximum}}$$

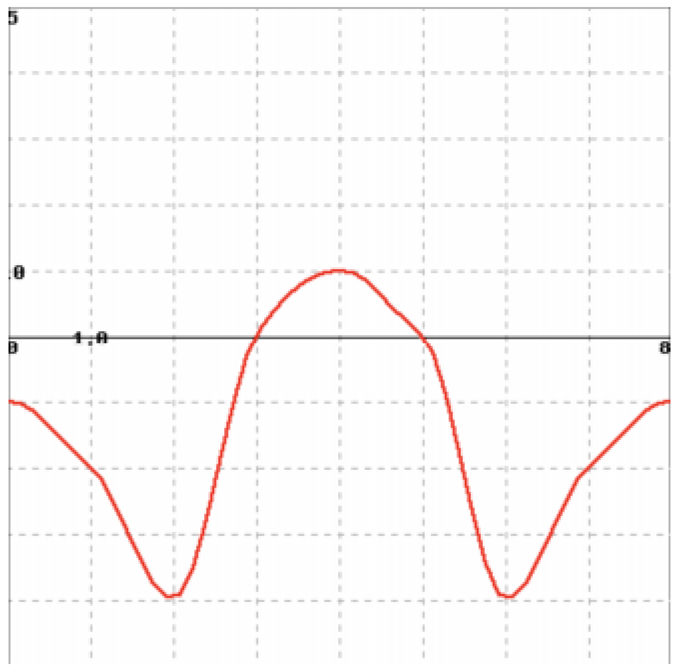
Helpful sign-chart sketch



f increasing $\to$ increasing $\to$ decreasing

Problem 11

We are given the graph of $f'(x)$ on the interval $(0, 8)$. Since the graph shown is the graph of the derivative, every conclusion about f must come from the behavior of $f'(x)$.



From the graph, we read the following key features:

- $f'(x) < 0$ on $(0, 3)$,
- $f'(x) > 0$ on $(3, 5)$,
- $f'(x) < 0$ again on $(5, 8)$,
- $f'(x)$ has a local minimum near $x = 2$,
- $f'(x)$ has a local maximum near $x = 4$,
- $f'(x)$ has another local minimum near $x = 6$.

These observations are enough to answer all parts.

(a) Where is $f(x)$ increasing?

A function f is increasing exactly where its derivative is positive:

$$f'(x) > 0.$$

Looking at the graph, the curve for $f'(x)$ lies above the x-axis only on the interval

$$(3, 5).$$

Therefore,

$$\boxed{f(x) \text{ is increasing on } (3, 5).}$$

Explanation

The derivative $f'(x)$ measures the slope of the graph of f. If $f'(x) > 0$, then the slope of f is positive, so f is rising. If $f'(x) < 0$, then f is falling.

Since the graph of $f'(x)$ is above the axis only between $x = 3$ and $x = 5$, that is the only interval where f is increasing.

(b) Where is $f(x)$ concave down?

A function f is concave down where

$$f''(x) < 0.$$

Since the graph given is $f'(x)$, this means f is concave down exactly where $f'(x)$ is decreasing.

From the graph, $f'(x)$ is decreasing on:

$$(0, 2) \quad \text{and} \quad (4, 6).$$

Therefore,

$$\boxed{f(x) \text{ is concave down on } (0, 2) \cup (4, 6).}$$

Explanation

Concavity tells us whether the slopes of f are increasing or decreasing.

- If $f'(x)$ is increasing, then the slopes of f are getting larger, so f is concave up.
- If $f'(x)$ is decreasing, then the slopes of f are getting smaller, so f is concave down.

On the graph, $f'(x)$ moves downward from $x = 0$ to $x = 2$, and again from $x = 4$ to $x = 6$. Those are exactly the intervals where f is concave down.

(c) Where does $f(x)$ have a local minimum?

A local minimum of f occurs where $f'(x)$ changes from negative to positive.

From the graph:

$$f'(x) < 0 \quad \text{for } x < 3, \qquad f'(x) > 0 \quad \text{for } x > 3 \text{ (near 3)}.$$

So $f'(x)$ changes sign from negative to positive at

$$x = 3.$$

Therefore,

$$\boxed{f(x) \text{ has a local minimum at } x = 3.}$$

Explanation

When $f'(x) < 0$, the function f is decreasing. When $f'(x) > 0$, the function f is increasing.

So if f changes from decreasing to increasing, it must have reached a low point. That is exactly what happens at $x = 3$.

(d) Where does $f(x)$ have inflection points?

An inflection point of f occurs where the concavity changes. Since concavity is determined by whether $f'(x)$ is increasing or decreasing, inflection points occur where $f'(x)$ has local extrema.

From the graph, $f'(x)$ has:

$$\text{a local minimum at } x = 2, \qquad \text{a local maximum at } x = 4, \qquad \text{a local minimum at } x = 6.$$

Therefore,

$$\boxed{f(x) \text{ has inflection points at } x = 2,\ 4,\ 6.}$$

Explanation

At each of these x-values, the graph of $f'(x)$ changes direction:

- near $x = 2$, $f'(x)$ changes from decreasing to increasing,
- near $x = 4$, $f'(x)$ changes from increasing to decreasing,
- near $x = 6$, $f'(x)$ changes from decreasing to increasing.

That means the concavity of f changes at each of those points, so those are inflection points.

Summary chart for Problem 11

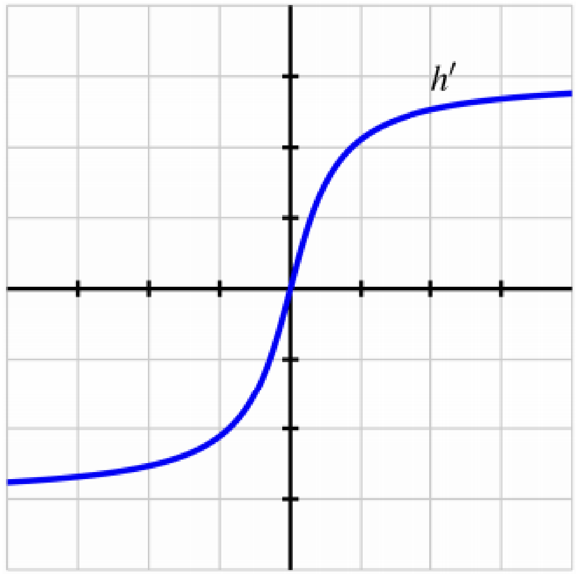
Question	Answer
Where is f increasing?	$(3, 5)$
Where is f concave down?	$(0, 2) \cup (4, 6)$
Where does f have a local minimum?	$x = 3$
Where does f have inflection points?	$x = 2, 4, 6$

A quick sign/shape summary is:

x	$(0, 2)$	2	$(2, 3)$	3	$(3, 4)$	4	$(4, 6)$ etc.
$f'(x)$	$-$	min	$-$	0	$+$	max	decreasing

Problem 12

We are given the graph of $h'(x)$, not $h(x)$. So every statement about h must be inferred from the sign and behavior of $h'(x)$.



From the graph, we observe:

- $h'(x) < 0$ for $x < 0$,
- $h'(0) = 0$,
- $h'(x) > 0$ for $x > 0$,
- $h'(x)$ is increasing throughout the graph shown.

We are also told in the problem that

$$\lim_{x \to \infty} h'(x) = 3.$$

(a) How many real solutions can $h(x) = 0$ have? Why?

Because

$$h'(x) < 0 \quad \text{for } x < 0,$$

the function h is decreasing on the left of 0.

Because

$$h'(x) > 0 \quad \text{for } x > 0,$$

the function h is increasing on the right of 0.

Since $h'(0) = 0$ and the sign changes from negative to positive, h has a minimum at $x = 0$.

A function with exactly one minimum can cross the x-axis at most twice. Depending on how high or low the minimum is, it may have:

- no real zeros,
- one real zero,
- or two real zeros.

So the correct statement is:

$$\boxed{h(x) = 0 \text{ can have at most 2 real solutions.}}$$

Explanation

The graph of $h'(x)$ shows that h goes down until $x = 0$, then goes up after $x = 0$. That means the graph of h has only one turning point, and that turning point is a minimum. A graph with one minimum cannot cross the x-axis more than twice.

(b) If $h(x) = 0$ has two distinct real solutions, what can you say about their signs?

If there are two distinct zeros, one must be to the left of the minimum and one must be to the right of the minimum.

Since the minimum occurs at $x = 0$, the two zeros must satisfy:

$$\boxed{\text{one zero is negative and the other is positive.}}$$

Explanation

On $(-\infty, 0)$, the function h is strictly decreasing, so it can cross the axis at most once there. On $(0, \infty)$, the function h is strictly increasing, so it can cross the axis at most once there.

Therefore, if two distinct zeros exist, they must occur on opposite sides of 0.

(c) If $\lim_{x\to\infty} h'(x) = 3$, how does the graph of $y = h(x)$ appear as $x \to \infty$?

The derivative $h'(x)$ represents the slope of the graph of $h(x)$. If

$$\lim_{x\to\infty} h'(x) = 3,$$

then for large x, the slope of h becomes very close to 3.

So as $x \to \infty$, the graph of $h(x)$

- keeps increasing,
- and looks more and more like a straight line with slope 3.

Thus,

$$\boxed{\text{As } x \to \infty,\ h(x) \text{ rises approximately linearly with slope 3.}}$$

Explanation

This does not necessarily mean the graph becomes exactly a line, but it means the tangent slopes approach 3, so visually the curve starts behaving like an increasing line whose steepness is about 3.

(d) Describe the concavity of $y = h(x)$ as fully as you can

Concavity of h depends on the sign of $h''(x)$. Since $h''(x)$ is the slope of $h'(x)$, we look at whether $h'(x)$ is increasing or decreasing.

From the graph, $h'(x)$ is increasing everywhere shown. Therefore

$$h''(x) > 0$$

everywhere shown.

So

$$\boxed{h(x) \text{ is concave up everywhere indicated by the graph.}}$$

Explanation

If $h'(x)$ increases, then the slopes of $h(x)$ are getting larger and larger. That is exactly what it means for h to be concave up.

There is no interval shown where $h'(x)$ decreases, so there is no evidence of any concave down interval.

Summary chart for Problem 12

Question	Answer
How many real zeros can $h(x) = 0$ have?	At most 2
If there are two distinct zeros, what are their signs?	One negative, one positive
How does $h(x)$ behave as $x \to \infty$?	Increasing, approximately linear with slope 3
Concavity of $h(x)$?	Concave up

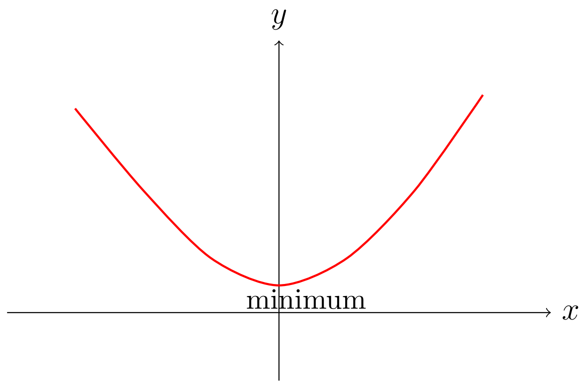
Helpful sign chart

x	$(-\infty, 0)$	0	$(0, \infty)$
$h'(x)$	$-$	0	$+$
$h(x)$	decreasing	minimum	increasing

Since $h'(x)$ is increasing throughout, $h(x)$ is concave up throughout.

Rough sketch of the shape of $h(x)$

The graph of $h(x)$ must look like a concave-up curve with one minimum at $x = 0$. A schematic sketch is:



This is not the exact graph of h, but it matches all the derivative information:

- decreasing for $x < 0$,
- increasing for $x > 0$,
- minimum at $x = 0$,
- concave up.