Detailed Solutions to Worksheet #1
MATH 031 – Spring 2026

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Exercise 1

Determine which equations are linear. If an equation is not linear, explain exactly why.

Hint

An equation is **linear** if:

- each variable appears only to the first power,
- variables are not multiplied together,
- variables do not appear inside nonlinear functions such as square roots, exponentials, trigonometric functions, etc.

(a) $x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 = 67$

Solution. This equation is **linear**.
Why? Each variable appears:

- only to the first power,
- with constant coefficients,
- and there are no products of variables or nonlinear functions.

(b) $x_1^2 = x_2 + x_3$

Solution. This equation is **not linear**.
Why? The term x_1^2 has exponent 2, so x_1 does not appear to the first power.

(c) $2x_1 - 3x_2 = 4x_5 - 8x_7 + 1$

Solution. This equation is **linear**.
Why? Even though variables appear on both sides, every variable is still first degree. We can rewrite it in standard linear form:

$$2x_1 - 3x_2 - 4x_5 + 8x_7 = 1.$$

This is a linear equation.

(d) $\sqrt{x_1} + x_2 = \sqrt{x_3} + 1$

Solution. This equation is **not linear**.

Why? Variables x_1 and x_3 appear inside square roots. Square root is a nonlinear operation.

(e) $e^{x_1} + x_2 = \cos(x_3)$

Solution. This equation is **not linear**.

Why? The variables x_1 and x_3 appear inside the nonlinear functions e^{x_1} and $\cos(x_3)$.

Final Answers for Exercise 1

(a) linear, (b) not linear, (c) linear, (d) not linear, (e) not linear.

Exercise 2

Solve each linear system and determine how many solutions it has.

General Hint

For a 2×2 system:

- if the two equations represent different lines intersecting once, there is **one solution**,
- if they represent the same line, there are **infinitely many solutions**,
- if they are parallel but distinct, there is **no solution**.

(a)

$$\begin{cases} 2x_1 - 7x_2 = -19, \\ -x_1 + 4x_2 = 11. \end{cases}$$

Step 1: Solve one equation for one variable.
From the second equation:

$$-x_1 + 4x_2 = 11$$

so

$$x_1 = 4x_2 - 11.$$

Step 2: Substitute into the first equation.

$$2(4x_2 - 11) - 7x_2 = -19.$$

Simplify:

$$8x_2 - 22 - 7x_2 = -19$$

$$x_2 - 22 = -19$$

$$x_2 = 3.$$

Step 3: Find x_1.

$$x_1 = 4(3) - 11 = 12 - 11 = 1.$$

Answer.

$$\boxed{(x_1, x_2) = (1, 3)}$$

There is **exactly one solution**.

(b)

$$\begin{cases} x_1 + 3x_2 = -2, \\ -3x_1 - 9x_2 = 6. \end{cases}$$

Step 1: Compare the equations.
If we multiply the first equation by -3, we get:

$$-3(x_1 + 3x_2) = -3(-2)$$

which gives

$$-3x_1 - 9x_2 = 6.$$

This is exactly the second equation.

Conclusion. Both equations describe the same line, so the system has **infinitely many solutions**.

Parametric form. Let $x_2 = t$. Then from the first equation,

$$x_1 + 3t = -2 \quad \Rightarrow \quad x_1 = -2 - 3t.$$

So all solutions are

$$\boxed{(x_1, x_2) = (-2 - 3t, t), \quad t \in \mathbb{R}.}$$

(c)

$$\begin{cases} 3x_1 - 4x_2 = 1, \\ 9x_1 - 12x_2 = 4. \end{cases}$$

Step 1: Compare the equations.
If we multiply the first equation by 3, we get:

$$3(3x_1 - 4x_2) = 3(1)$$

so

$$9x_1 - 12x_2 = 3.$$

But the second equation is

$$9x_1 - 12x_2 = 4.$$

Step 2: Interpret.
The left-hand sides are the same, but the right-hand sides are different. That means the system is inconsistent.

Answer.

$$\boxed{\text{No solution}}$$

So the system has **zero solutions**.

Final Answers for Exercise 2

$$\boxed{\begin{aligned} &\text{(a) one solution } (1,3), \\ &\text{(b) infinitely many solutions}, \\ &\text{(c) no solution.} \end{aligned}}$$

Exercise 3

Write the coefficient matrix and augmented matrix for each system.

Hint

- The **coefficient matrix** contains only the coefficients of the variables.
- The **augmented matrix** adds the constants on the right as one extra column.
- If a variable is missing from an equation, its coefficient is 0.

(a)

$$\begin{cases} x_1 + 3x_2 - 2x_3 = 4, \\ 4x_1 - 2x_2 + x_3 = 1, \\ -x_1 + 2x_2 + 4x_3 = 0. \end{cases}$$

Coefficient matrix:

$$\boxed{A = \begin{bmatrix} 1 & 3 & -2 \\ 4 & -2 & 1 \\ -1 & 2 & 4 \end{bmatrix}}$$

Augmented matrix:

$$\boxed{\left[\begin{array}{ccc|c} 1 & 3 & -2 & 4 \\ 4 & -2 & 1 & 1 \\ -1 & 2 & 4 & 0 \end{array}\right]}$$

(b)

$$\begin{cases} x_1 + x_2 = 1, \\ x_1 + 2x_2 = 2, \\ x_1 + 3x_2 = 3, \\ x_1 + 4x_2 = 10. \end{cases}$$

Coefficient matrix:

$$\boxed{A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}}$$

Augmented matrix:

$$\boxed{\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 3 & 3 \\ 1 & 4 & 10 \end{array}\right]}$$

(c)

$$\begin{cases} x_1 + x_3 + x_5 + x_6 = 1, \\ x_1 + x_2 - x_4 - x_6 = -1. \end{cases}$$

The variables are $x_1, x_2, x_3, x_4, x_5, x_6$.

Caution: A common mistake is to forget the missing variables and you forget the coefficient 0 for those variables. For example, x_2, x_4 are missing in the first equation.

First equation:

$$x_1 + 0x_2 + x_3 + 0x_4 + x_5 + x_6 = 1$$

Second equation:

$$x_1 + x_2 + 0x_3 - x_4 + 0x_5 - x_6 = -1$$

Coefficient matrix:

$$\boxed{A = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & -1 & 0 & -1 \end{bmatrix}}$$

Augmented matrix:

$$\boxed{\left[\begin{array}{cccccc|c} 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & -1 & 0 & -1 & -1 \end{array}\right]}$$

Exercise 4

Convert systems between matrix form and equation form.

(a)

$$\begin{bmatrix} 3 & 1 & 5 & -2 \\ 4 & -5 & 1 & 2 \\ 5 & -3 & -5 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}.$$

Hint

Each row of the matrix gives one equation.

Row 1:

$$3x_1 + x_2 + 5x_3 - 2x_4 = 0$$

Row 2:

$$4x_1 - 5x_2 + x_3 + 2x_4 = 1$$

Row 3:

$$5x_1 - 3x_2 - 5x_3 + 9x_4 = 2$$

Answer:

$$\boxed{\begin{cases} 3x_1 + x_2 + 5x_3 - 2x_4 = 0, \\ 4x_1 - 5x_2 + x_3 + 2x_4 = 1, \\ 5x_1 - 3x_2 - 5x_3 + 9x_4 = 2. \end{cases}}$$

(b)

$$\begin{cases} -4x_1 + x_3 = 0, \\ 2x_1 + x_2 = 7, \\ x_2 - 3x_3 = 2. \end{cases}$$

Hint

We list the coefficients of x_1, x_2, x_3 in each equation.

Equation 1: $-4x_1 + 0x_2 + x_3 = 0$
Equation 2: $2x_1 + x_2 + 0x_3 = 7$
Equation 3: $0x_1 + x_2 - 3x_3 = 2$
So

$$\boxed{\begin{bmatrix} -4 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 7 \\ 2 \end{bmatrix}}$$

Exercise 5

Find for which values of a and b the systems are consistent.

(a)

$$\begin{cases} x_1 + ax_2 = 1, \\ -x_1 + 3x_2 = b. \end{cases}$$

Hint

A system is **consistent** if it has at least one solution. For a 2×2 system, we usually check whether the determinant of the coefficient matrix is zero or not.

The coefficient matrix is

$$A = \begin{bmatrix} 1 & a \\ -1 & 3 \end{bmatrix}.$$

Its determinant is

$$\det(A) = 1 \cdot 3 - (-1) \cdot a = 3 + a.$$

Case 1: $a \neq -3$

Then

$$\det(A) \neq 0,$$

so the system has a unique solution. Therefore it is consistent for *every* b.

Case 2: $a = -3$

Then the system becomes

$$\begin{cases} x_1 - 3x_2 = 1, \\ -x_1 + 3x_2 = b. \end{cases}$$

Multiply the first equation by -1:

$$-x_1 + 3x_2 = -1.$$

To be consistent, this must match the second equation, so we need

$$b = -1.$$

If $b \neq -1$, the system is inconsistent.

Conclusion for (a)

$$\boxed{\text{The system is consistent for all } (a, b) \text{ except when } a = -3 \text{ and } b \neq -1.}$$

Equivalently:

$$\boxed{\text{consistent if } a \neq -3 \text{ for any } b, \quad \text{or if } a = -3 \text{ and } b = -1.}$$

Exercise 5(a) — Solution without Determinants

We are given the system:

$$\begin{cases} x_1 + ax_2 = 1, \\ -x_1 + 3x_2 = b. \end{cases}$$

Step 1: Add the two equations

Add the equations to eliminate x_1:

$$(x_1 + ax_2) + (-x_1 + 3x_2) = 1 + b.$$

Simplify:

$$(a + 3)x_2 = 1 + b.$$

Step 2: Analyze cases

Case 1: $a + 3 \neq 0$ (i.e., $a \neq -3$)
We can solve for x_2:

$$x_2 = \frac{1+b}{a+3}.$$

Then substitute back into the first equation:

$$x_1 = 1 - ax_2 = 1 - a\left(\frac{1+b}{a+3}\right).$$

So there is a **unique solution**. Hence, the system is **consistent for all b when** $a \neq -3$.

Case 2: $a + 3 = 0$ (i.e., $a = -3$)
Then the equation becomes:

$$0 \cdot x_2 = 1 + b,$$

which simplifies to:

$$0 = 1 + b.$$

Subcase: $b = -1$
Then $0 = 0$, so the system is consistent. There are infinitely many solutions.
Subcase: $b \neq -1$
Then we get a contradiction:

$$0 = 1 + b \neq 0,$$

so the system is **inconsistent** (no solution).

Final Answer

$$\boxed{\text{The system is consistent for all } a \neq -3 \text{ (any } b\text{)}, \quad \text{and also when } a = -3 \text{ and } b = -1.}$$

(b)

$$\begin{cases} x_1 + x_2 + ax_3 = 1, \\ -x_1 + 2x_2 = b, \\ x_2 + x_3 = 0. \end{cases}$$

Hint

Use substitution.

Step 1: Solve the third equation for x_3.

$$x_2 + x_3 = 0 \quad \Rightarrow \quad x_3 = -x_2.$$

Step 2: Solve the second equation for x_1.

$$-x_1 + 2x_2 = b \quad \Rightarrow \quad x_1 = 2x_2 - b.$$

Step 3: Substitute into the first equation.

$$x_1 + x_2 + ax_3 = 1.$$

Substitute $x_1 = 2x_2 - b$ and $x_3 = -x_2$:

$$(2x_2 - b) + x_2 + a(-x_2) = 1.$$

Simplify:

$$\begin{gathered} 2x_2 - b + x_2 - ax_2 = 1 \\ (3-a)x_2 - b = 1 \\ (3-a)x_2 = 1 + b. \end{gathered}$$

Now discuss cases.

Case 1: $a \neq 3$

Then $3 - a \neq 0$, so

$$x_2 = \frac{1+b}{3-a}.$$

Then x_1 and x_3 are determined, so the system has a unique solution. Hence it is consistent for every b.

Case 2: $a = 3$

Then the equation becomes

$$(3-3)x_2 = 1 + b$$

so

$$0 = 1 + b.$$

Thus:

- if $b = -1$, the equation is $0 = 0$, so the system is consistent,
- if $b \neq -1$, the equation is impossible, so the system is inconsistent.

Conclusion for (b)

$$\boxed{\text{The system is consistent for all } (a,b) \text{ except when } a = 3 \text{ and } b \neq -1.}$$

Equivalently:

$$\boxed{\text{consistent if } a \neq 3 \text{ for any } b, \quad \text{or if } a = 3 \text{ and } b = -1.}$$

Summary of Exercise 5

$$\boxed{\begin{array}{ll} \text{(a)} & \text{consistent except when } a = -3,\ b \neq -1, \\ \text{(b)} & \text{consistent except when } a = 3,\ b \neq -1. \end{array}}$$