

Detailed Solutions to Worksheet #2

MATH 031 – Spring 2026

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Exercise 1

Consider each matrix as the augmented matrix of a linear system, and state in words the next two elementary row operations that should be performed in order to solve the system.

Hint

The goal is to continue reducing the matrix toward reduced row echelon form (RREF). Look for pivot entries and eliminate entries above or below them.

(i)

$$\left[\begin{array}{cccc|c} 1 & -4 & 5 & 0 & 7 \\ 0 & -1 & -3 & 0 & 6 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -5 \end{array} \right]$$

Step 1: Use the pivot in row 3 to eliminate the entry above it in row 2. Since row 2 has -3 in column 3, add 3 times row 3 to row 2:

$$R_2 \leftarrow R_2 + 3R_3.$$

Step 2: Use the pivot in row 3 to eliminate the entry above it in row 1. Since row 1 has 5 in column 3, subtract 5 times row 3 from row 1:

$$R_1 \leftarrow R_1 - 5R_3.$$

Answer in words:

- Add 3 times row 3 to row 2.
- Subtract 5 times row 3 from row 1.

(ii)

$$\left[\begin{array}{cccc|c} 1 & -6 & 4 & 0 & -1 \\ 0 & 2 & -7 & 0 & 4 \\ 0 & 0 & 1 & 2 & -3 \\ 0 & 0 & 3 & 1 & 6 \end{array} \right]$$

Step 1: Use row 3 to eliminate the 3 in row 4, column 3:

$$R_4 \leftarrow R_4 - 3R_3.$$

Let us compute:

$$\begin{aligned} R_4 - 3R_3 &= [0 \ 0 \ 3 \ 1 \ | \ 6] - 3[0 \ 0 \ 1 \ 2 \ | \ -3] \\ &= [0 \ 0 \ 0 \ -5 \ | \ 15]. \end{aligned}$$

Step 2: Make the leading entry in row 4 equal to 1 by dividing row 4 by -5 :

$$R_4 \leftarrow -\frac{1}{5}R_4.$$

Answer in words:

- Subtract 3 times row 3 from row 4.
- Multiply row 4 by $-\frac{1}{5}$.

Exercise 2

In each case, continue the row operations and describe the solution set of the original system.

Hint

Interpret each augmented matrix as a system. Continue row reduction until you can clearly see whether:

- the system has no solution,
- exactly one solution,
- or infinitely many solutions.

(i)

$$\left[\begin{array}{ccc|c} 1 & 7 & 3 & -4 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array} \right]$$

Observation: The third row says

$$0x_1 + 0x_2 + 0x_3 = 1,$$

that is,

$$0 = 1,$$

which is impossible.

Therefore the system is **inconsistent**.

Solution set:

$$\boxed{\emptyset}$$

(ii)

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & 7 & 0 \\ 0 & 0 & 2 & -2 \end{array} \right]$$

Step 1: Make the pivot in row 3 equal to 1:

$$R_3 \leftarrow \frac{1}{2}R_3.$$

Then

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & 7 & 0 \\ 0 & 0 & 1 & -1 \end{array} \right]$$

Step 2: Eliminate the entries above the pivot in column 3.

From row 2:

$$R_2 \leftarrow R_2 - 7R_3$$

gives

$$[0 \ 1 \ 0 \ | \ 7].$$

From row 1:

$$R_1 \leftarrow R_1 - 2R_3$$

gives

$$[1 \ 1 \ 0 \ | \ 2].$$

So now we have

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -1 \end{array} \right]$$

Step 3: Eliminate the 1 above the pivot in column 2:

$$R_1 \leftarrow R_1 - R_2$$

so

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -1 \end{array} \right]$$

Thus the solution is

$$\boxed{x_1 = -5, \quad x_2 = 7, \quad x_3 = -1.}$$

The system has **one unique solution**.

(iii)

$$\left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & -4 \\ 0 & 1 & -3 & 0 & -7 \\ 0 & 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 & 4 \end{array} \right]$$

The last row says

$$0 = 4,$$

which is impossible.

Therefore the system is **inconsistent**.

Solution set:

$$\boxed{\emptyset}$$

Exercise 3

Solve the following systems.

Hint

Write the augmented matrix and use row operations carefully. After obtaining echelon form or RREF, interpret the solution.

(a)

$$\begin{cases} x_2 + 4x_3 = -4, \\ x_1 + 3x_2 + 3x_3 = -2, \\ 3x_1 + 7x_2 + 5x_3 = 6. \end{cases}$$

Step 1: Write the augmented matrix

$$\left[\begin{array}{ccc|c} 0 & 1 & 4 & -4 \\ 1 & 3 & 3 & -2 \\ 3 & 7 & 5 & 6 \end{array} \right]$$

Swap row 1 and row 2 to place a pivot in the first column:

$$R_1 \leftrightarrow R_2$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 3 & -2 \\ 0 & 1 & 4 & -4 \\ 3 & 7 & 5 & 6 \end{array} \right]$$

Step 2: Eliminate below the pivot in column 1

$$R_3 \leftarrow R_3 - 3R_1$$

$$[3 \ 7 \ 5 \ | \ 6] - 3[1 \ 3 \ 3 \ | \ -2] = [0 \ -2 \ -4 \ | \ 12].$$

So we get

$$\left[\begin{array}{ccc|c} 1 & 3 & 3 & -2 \\ 0 & 1 & 4 & -4 \\ 0 & -2 & -4 & 12 \end{array} \right]$$

Step 3: Eliminate below the pivot in column 2

$$R_3 \leftarrow R_3 + 2R_2$$

$$[0 \ -2 \ -4 \ | \ 12] + 2[0 \ 1 \ 4 \ | \ -4] = [0 \ 0 \ 4 \ | \ 4].$$

Thus

$$\left[\begin{array}{ccc|c} 1 & 3 & 3 & -2 \\ 0 & 1 & 4 & -4 \\ 0 & 0 & 4 & 4 \end{array} \right]$$

Step 4: Solve by back-substitution

From row 3:

$$4x_3 = 4 \Rightarrow x_3 = 1.$$

From row 2:

$$x_2 + 4(1) = -4 \Rightarrow x_2 = -8.$$

From row 1:

$$x_1 + 3(-8) + 3(1) = -2$$

$$x_1 - 24 + 3 = -2$$

$$x_1 - 21 = -2$$

$$x_1 = 19.$$

Answer:

$$\boxed{(x_1, x_2, x_3) = (19, -8, 1)}$$

(b)

$$\begin{cases} x_1 - 3x_2 + 4x_3 = -4, \\ 3x_1 - 7x_2 + 7x_3 = -8, \\ -4x_1 + 6x_2 + 2x_3 = 4. \end{cases}$$

Step 1: Write the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & -3 & 4 & -4 \\ 3 & -7 & 7 & -8 \\ -4 & 6 & 2 & 4 \end{array} \right]$$

Step 2: Eliminate below the pivot in column 1

$$R_2 \leftarrow R_2 - 3R_1$$

$$[3 \ -7 \ 7 \ | \ -8] - 3[1 \ -3 \ 4 \ | \ -4] = [0 \ 2 \ -5 \ | \ 4].$$

$$R_3 \leftarrow R_3 + 4R_1$$

$$[-4 \ 6 \ 2 \ | \ 4] + 4[1 \ -3 \ 4 \ | \ -4] = [0 \ -6 \ 18 \ | \ -12].$$

So

$$\left[\begin{array}{ccc|c} 1 & -3 & 4 & -4 \\ 0 & 2 & -5 & 4 \\ 0 & -6 & 18 & -12 \end{array} \right]$$

Step 3: Eliminate below the pivot in column 2

A convenient choice is

$$R_3 \leftarrow R_3 + 3R_2$$

since

$$[0 \ -6 \ 18 \ | \ -12] + 3[0 \ 2 \ -5 \ | \ 4] = [0 \ 0 \ 3 \ | \ 0].$$

Thus

$$\left[\begin{array}{ccc|c} 1 & -3 & 4 & -4 \\ 0 & 2 & -5 & 4 \\ 0 & 0 & 3 & 0 \end{array} \right]$$

Step 4: Back-substitution

From row 3:

$$3x_3 = 0 \quad \Rightarrow \quad x_3 = 0.$$

From row 2:

$$2x_2 - 5(0) = 4 \quad \Rightarrow \quad x_2 = 2.$$

From row 1:

$$x_1 - 3(2) + 4(0) = -4$$

$$x_1 - 6 = -4$$

$$x_1 = 2.$$

Answer:

$$\boxed{(x_1, x_2, x_3) = (2, 2, 0)}$$

Exercise 4

(a)

Do the three lines

$$x_1 - 4x_2 = 1, \quad 2x_1 - x_2 = -3, \quad -x_1 - 3x_2 = 4$$

have a common point of intersection in the x_1x_2 -plane?

Hint

If three lines have a common point, then that point must satisfy all three equations. Solve two equations first, then check the third.

Step 1: Solve the first two equations

$$\begin{cases} x_1 - 4x_2 = 1 \\ 2x_1 - x_2 = -3 \end{cases}$$

From the first equation:

$$x_1 = 1 + 4x_2.$$

Substitute into the second:

$$2(1 + 4x_2) - x_2 = -3$$

$$2 + 8x_2 - x_2 = -3$$

$$2 + 7x_2 = -3$$

$$7x_2 = -5$$

$$x_2 = -\frac{5}{7}.$$

Then

$$x_1 = 1 + 4\left(-\frac{5}{7}\right) = 1 - \frac{20}{7} = -\frac{13}{7}.$$

Step 2: Check the third line

Substitute into

$$-x_1 - 3x_2 = 4.$$

We get

$$-\left(-\frac{13}{7}\right) - 3\left(-\frac{5}{7}\right) = \frac{13}{7} + \frac{15}{7} = \frac{28}{7} = 4.$$

So the point satisfies the third equation too.

Conclusion: Yes, the three lines have a common point of intersection:

$$\boxed{\left(-\frac{13}{7}, -\frac{5}{7}\right)}.$$

(b)

Do the three planes

$$x_1 + 2x_2 + x_3 = 4, \quad x_2 - x_3 = 1, \quad x_1 + 3x_2 = 0$$

have at least one common point of intersection in $x_1x_2x_3$ -space?

Hint

Solve the system and see whether it is consistent.

We solve:

$$\begin{cases} x_1 + 2x_2 + x_3 = 4 \\ x_2 - x_3 = 1 \\ x_1 + 3x_2 = 0 \end{cases}$$

From the second equation:

$$x_2 = 1 + x_3.$$

From the third equation:

$$x_1 = -3x_2.$$

Substitute both into the first:

$$-3x_2 + 2x_2 + x_3 = 4$$

$$-x_2 + x_3 = 4.$$

Now substitute $x_2 = 1 + x_3$:

$$-(1 + x_3) + x_3 = 4$$

$$-1 = 4,$$

which is impossible.

Therefore the system is inconsistent.

Conclusion:

No, the three planes do not have a common point.

Exercise 5

True or false? Briefly justify your answers.

(i) Every elementary row operation is reversible.

Answer: True.

Reason: Each elementary row operation has an inverse operation:

- Swapping two rows is undone by swapping them back.
- Multiplying a row by a nonzero constant is undone by multiplying by its reciprocal.
- Adding a multiple of one row to another is undone by subtracting the same multiple.

(ii) Elementary row operations on an augmented matrix never change the solution set of the associated linear systems.

Answer: True.

Reason: Elementary row operations produce an equivalent system. Equivalent systems have exactly the same solution set.

(iii) A 5×6 matrix has six rows.

Answer: False.

Reason: A 5×6 matrix has 5 rows and 6 columns.

(iv) Two matrices are row-equivalent if they have the same number of rows.

Answer: False.

Reason: Having the same number of rows is not enough. Two matrices are row-equivalent if one can be obtained from the other by a sequence of elementary row operations.

(v) An inconsistent system has more than one solution.

Answer: False.

Reason: An inconsistent system has **no solution**.

(vi) Two linear systems are equivalent if they have the same solution set.

Answer: True.

Reason: This is exactly the definition of equivalent systems.

Final Answers Summary

- | |
|-------------|
| (i) True |
| (ii) True |
| (iii) False |
| (iv) False |
| (v) False |
| (vi) True |

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