MATH031 – SPRING 2026
Worksheet #3 Detailed Solutions

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Worksheet source: MATH031 – Spring 2026 Worksheet #3. The exercises solved below are based on the worksheet.

Exercise 1

Describe all possible echelon forms for nonzero matrices of sizes 2×2 and 3×3, using the symbols $\blacksquare$, $*$, and 0.

Here:

$$\blacksquare = \text{pivot entry}, \qquad * = \text{any number}, \qquad 0 = \text{zero}.$$

Recall the rules for echelon form:

1. All nonzero rows must be above any zero rows.
2. In each nonzero row, the first nonzero entry is called a pivot.
3. As you move downward, each pivot must be to the right of the pivot above it.
4. All entries below a pivot must be zero.

So to list all possible echelon forms, we only need to list all possible pivot locations.

2×2 matrices

A nonzero 2×2 matrix must have either one nonzero row or two nonzero rows.

Case 1: Two nonzero rows.

Then the first row has a pivot, and the second row must also have a pivot to the right of the first one. So the first pivot must be in column 1, and the second pivot must be in column 2. Thus the form is

$$\begin{bmatrix} \blacksquare & * \\ 0 & \blacksquare \end{bmatrix}.$$

Case 2: One nonzero row.

Then the second row must be all zeros. The first row may have its pivot in column 1 or in column 2.

If the pivot is in column 1, the form is

$$\begin{bmatrix} \blacksquare & * \\ 0 & 0 \end{bmatrix}.$$

If the pivot is in column 2, the form is

$$\begin{bmatrix} 0 & \blacksquare \\ 0 & 0 \end{bmatrix}.$$

Therefore all possible echelon forms for nonzero 2×2 matrices are

$$\boxed{\begin{bmatrix} \blacksquare & * \\ 0 & \blacksquare \end{bmatrix}, \quad \begin{bmatrix} \blacksquare & * \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & \blacksquare \\ 0 & 0 \end{bmatrix}.}$$

3×3 matrices

A nonzero 3×3 matrix can have one nonzero row, two nonzero rows, or three nonzero rows.

Case 1: Three nonzero rows.

Then each row has a pivot. Since pivots must move to the right as we go down, the pivots must appear in columns 1, 2, and 3. So the only possibility is

$$\begin{bmatrix} \blacksquare & * & * \\ 0 & \blacksquare & * \\ 0 & 0 & \blacksquare \end{bmatrix}.$$

Case 2: Two nonzero rows.

Then the third row must be all zeros. We place pivots in the first two rows so that the second pivot is to the right of the first.

There are three possibilities:

- First pivot in column 1, second pivot in column 2:

$$\begin{bmatrix} \blacksquare & * & * \\ 0 & \blacksquare & * \\ 0 & 0 & 0 \end{bmatrix}$$

- First pivot in column 1, second pivot in column 3:

$$\begin{bmatrix} \blacksquare & * & * \\ 0 & 0 & \blacksquare \\ 0 & 0 & 0 \end{bmatrix}$$

- First pivot in column 2, second pivot in column 3:

$$\begin{bmatrix} 0 & \blacksquare & * \\ 0 & 0 & \blacksquare \\ 0 & 0 & 0 \end{bmatrix}$$

Case 3: One nonzero row.

Then only the first row is nonzero, and the pivot can be in column 1, 2, or 3.

So the possibilities are

$$\begin{bmatrix} \blacksquare & * & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & \blacksquare & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & \blacksquare \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Therefore all possible echelon forms for nonzero 3×3 matrices are

$$\boxed{\begin{bmatrix} \blacksquare & * & * \\ 0 & \blacksquare & * \\ 0 & 0 & \blacksquare \end{bmatrix}, \quad \begin{bmatrix} \blacksquare & * & * \\ 0 & \blacksquare & * \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} \blacksquare & * & * \\ 0 & 0 & \blacksquare \\ 0 & 0 & 0 \end{bmatrix},}$$

$$\boxed{\begin{bmatrix} 0 & \blacksquare & * \\ 0 & 0 & \blacksquare \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} \blacksquare & * & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & \blacksquare & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & \blacksquare \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.}$$

Final Answer

So the possible echelon forms are exactly those obtained by placing pivots so that:

- each pivot is to the right of the pivot above it,
- everything below each pivot is zero,
- and all zero rows are at the bottom.

Exercise 2

Solve the system

$$\begin{cases} x^2 - y^3 + 3z^4 = 4, \\ 2x^2 - 3y^3 + 5z^4 = -1, \\ -x^2 + 2y^3 - 2z^4 = 5. \end{cases}$$

The worksheet says this is a nonlinear system and asks how we can use ideas from class to solve it. The key idea is to introduce new variables so that the system becomes linear.

Step 1: Substitute new variables

Let

$$u = x^2, \qquad v = y^3, \qquad w = z^4.$$

Then the system becomes

$$\begin{cases} u - v + 3w = 4, \\ 2u - 3v + 5w = -1, \\ -u + 2v - 2w = 5. \end{cases}$$

Now this is a linear system in the variables u, v, w.

Step 2: Solve the linear system

Write the augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & -1 & 3 & 4 \\ 2 & -3 & 5 & -1 \\ -1 & 2 & -2 & 5 \end{array}\right].$$

We row reduce.

First, eliminate entries below the pivot in column 1:

$$R_2 \leftarrow R_2 - 2R_1, \qquad R_3 \leftarrow R_3 + R_1.$$

This gives

$$\left[\begin{array}{ccc|c} 1 & -1 & 3 & 4 \\ 0 & -1 & -1 & -9 \\ 0 & 1 & 1 & 9 \end{array}\right].$$

Now add row 2 and row 3:

$$R_3 \leftarrow R_3 + R_2,$$

so

$$\left[\begin{array}{ccc|c} 1 & -1 & 3 & 4 \\ 0 & -1 & -1 & -9 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

This means the system has infinitely many solutions in the variables u, v, w, with one free variable.

Step 3: Express the solution parametrically

From row 2:

$$-v - w = -9 \implies v = -w + 9.$$

From row 1:

$$u - v + 3w = 4.$$

Substitute $v = -w + 9$:

$$\begin{aligned} u - (-w + 9) + 3w &= 4 \\ u + w - 9 + 3w &= 4 \\ u + 4w &= 13 \\ u &= 13 - 4w. \end{aligned}$$

Let

$$w = t.$$

Then

$$u = 13 - 4t, \qquad v = 9 - t, \qquad w = t.$$

So

$$x^2 = 13 - 4t, \qquad y^3 = 9 - t, \qquad z^4 = t.$$

Step 4: Translate back to x, y, z

Because $x^2 \geq 0$ and $z^4 \geq 0$, we must have

$$t \geq 0, \qquad 13 - 4t \geq 0.$$

Thus

$$0 \leq t \leq \frac{13}{4}.$$

Also,

$$y = \sqrt[3]{9 - t},$$

since cube roots are defined for all real numbers.

Hence the full real solution set is

$$\boxed{x = \pm\sqrt{13-4t}, \qquad y = \sqrt[3]{9-t}, \qquad z = \pm t^{1/4}, \qquad 0 \le t \le \frac{13}{4}.}$$

Conclusion

So the system as written actually has infinitely many real solutions, not just four.

If one wants exactly four solutions, that would happen only for special choices of t where both x and z are nonzero and each contributes a $\pm$ choice while y is determined uniquely. But for the equations printed here, t is free, so there are infinitely many real solutions.

Exercise 3

Find the general solutions of the systems whose augmented matrices are given. If they are consistent, give the solutions parametrically in vector form.

(a)

The augmented matrix is

$$\begin{bmatrix} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ -1 & 7 & -4 & 2 & 7 \end{bmatrix}.$$

This corresponds to a system with 4 variables, say x_1, x_2, x_3, x_4.

Step 1: Row reduce

Start with

$$\left[\begin{array}{cccc|c} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ -1 & 7 & -4 & 2 & 7 \end{array}\right].$$

Add row 1 to row 3:

$$R_3 \leftarrow R_3 + R_1.$$

Then

$$\left[\begin{array}{cccc|c} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ 0 & 0 & -4 & 8 & 12 \end{array}\right].$$

Now eliminate the entry in row 3, column 3:

$$R_3 \leftarrow R_3 + 4R_2.$$

This gives

$$\left[\begin{array}{cccc|c} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

So the system is consistent.

Step 2: Identify pivot and free variables

Pivot columns are columns 1 and 3. Therefore:

$$x_1,\ x_3 \text{ are basic variables,} \qquad x_2,\ x_4 \text{ are free variables.}$$

Let

$$x_2 = s, \qquad x_4 = t.$$

Step 3: Solve for the basic variables

From row 2:

$$x_3 - 2x_4 = -3 \quad \Longrightarrow \quad x_3 = -3 + 2t.$$

From row 1:

$$x_1 - 7x_2 + 6x_4 = 5 \quad \Longrightarrow \quad x_1 = 5 + 7s - 6t.$$

So

$$x_1 = 5 + 7s - 6t, \qquad x_2 = s, \qquad x_3 = -3 + 2t, \qquad x_4 = t.$$

Step 4: Write in vector form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ -3 \\ 0 \end{bmatrix} + s\begin{bmatrix} 7 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -6 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \qquad s, t \in \mathbb{R}.$$

Therefore the general solution is

$$\boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ -3 \\ 0 \end{bmatrix} + s\begin{bmatrix} 7 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -6 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \qquad s, t \in \mathbb{R}.}$$

(b)

The augmented matrix is

$$\begin{bmatrix} 1 & -3 & 0 & -1 & 0 & 2 \\ 0 & 1 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 1 & 9 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

This corresponds to a system with 5 variables, say x_1, x_2, x_3, x_4, x_5.

Step 1: Read off the equations

The rows correspond to

$$x_1 - 3x_2 - x_4 = 2,$$
$$x_2 - 4x_5 = 1,$$
$$x_4 + 9x_5 = -4.$$

Step 2: Identify pivot and free variables

Pivot columns are columns 1, 2, and 4. So

$$x_1,\ x_2,\ x_4 \text{ are basic,} \qquad x_3,\ x_5 \text{ are free.}$$

Let

$$x_3 = s, \qquad x_5 = t.$$

Step 3: Solve for the basic variables

From row 2:

$$x_2 - 4x_5 = 1 \quad \Longrightarrow \quad x_2 = 1 + 4t.$$

From row 3:

$$x_4 + 9x_5 = -4 \quad \Longrightarrow \quad x_4 = -4 - 9t.$$

From row 1:

$$x_1 - 3x_2 - x_4 = 2.$$

Substitute x_2 and x_4:

$$x_1 - 3(1 + 4t) - (-4 - 9t) = 2.$$

Simplify:

$$x_1 - 3 - 12t + 4 + 9t = 2$$
$$x_1 + 1 - 3t = 2$$
$$x_1 = 1 + 3t.$$

So

$$x_1 = 1 + 3t, \qquad x_2 = 1 + 4t, \qquad x_3 = s, \qquad x_4 = -4 - 9t, \qquad x_5 = t.$$

Step 4: Write in vector form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -4 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ 4 \\ 0 \\ -9 \\ 1 \end{bmatrix}, \qquad s, t \in \mathbb{R}.$$

Therefore the general solution is

$$\boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -4 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ 4 \\ 0 \\ -9 \\ 1 \end{bmatrix}, \qquad s, t \in \mathbb{R}.}$$

Exercise 4

"Soft" questions.

(a)

Suppose a 3×5 coefficient matrix for a system has three pivot columns. Must the system be consistent? Why?

Answer: No, not necessarily.

Explanation: The coefficient matrix by itself only tells us about the coefficients of the variables. Consistency depends on the *augmented matrix*, because the constants on the right-hand side matter too.

A system is inconsistent exactly when row reduction produces a row like

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \big| & b \end{bmatrix} \quad \text{with } b \neq 0.$$

It is possible for the coefficient matrix to have three pivots, but when the constants are included, the augmented matrix may still produce such a contradictory row.

So having three pivot columns in the coefficient matrix does *not* by itself guarantee consistency.

$$\boxed{\text{No, the system need not be consistent.}}$$

(b)

Suppose a 3×5 augmented matrix has a pivot in the fifth column. Must the system be consistent? Why?

Answer: No.

Explanation: In a 3×5 augmented matrix, the fifth column is the augmented column, meaning it contains the constants on the right-hand side. If there is a pivot in the fifth column, then after row reduction we can get a row of the form

$$\begin{bmatrix} 0 & 0 & 0 & 0 & \big| & 1 \end{bmatrix},$$

which corresponds to the false equation

$$0 = 1.$$

That means the system is inconsistent.

So in fact, a pivot in the last column is a signal of inconsistency.

$$\boxed{\text{No. A pivot in the augmented column means the system is inconsistent.}}$$

(c)

Suppose the coefficient matrix of a linear system has a pivot position in every row. Explain why the system must be consistent.

Answer: Yes, the system must be consistent.

Explanation: If the coefficient matrix has a pivot in every row, then no row of the coefficient matrix becomes all zeros. Therefore, when we form the augmented matrix and row reduce, we cannot get a row of the form

$$\begin{bmatrix} 0 & 0 & \cdots & 0 & | & b \end{bmatrix} \quad \text{with } b \neq 0,$$

because such a row would mean the coefficient row is all zero.

Since inconsistency only occurs when such a contradictory row appears, and that cannot happen here, the system must be consistent.

$$\boxed{\text{Yes, a pivot in every row of the coefficient matrix guarantees consistency.}}$$

Exercise 5

True or false? Briefly justify your answers.

(i)

The echelon form of a matrix is unique.

Answer: False.

Explanation: A matrix can have different echelon forms depending on the row operations used. For example, different multiples of rows may lead to different echelon-form entries. What *is* unique is the reduced row echelon form (RREF), not just any echelon form.

$$\boxed{\text{False}}$$

(ii)

The pivot positions in a matrix depend on whether row interchanges are used in the row reduction process.

Answer: False.

Explanation: The pivot positions are determined by the matrix itself. Row interchanges may change the order in which we see the pivots during computation, but the pivot columns in the original matrix are intrinsic and do not depend on the method used. Equivalent row reduction processes lead to the same pivot positions.

$$\boxed{\text{False}}$$

(iii)

A basic variable in a linear system is a variable that corresponds to a pivot column in the coefficient matrix.

Answer: True.

Explanation: By definition, basic variables correspond to pivot columns, while variables corresponding to nonpivot columns are free variables.

$$\boxed{\text{True}}$$

(iv)

Finding a parametric description of the solution set of a linear system is the same as solving it.

Answer: True.

Explanation: Solving a linear system means describing all solutions. When a system has infinitely many solutions, the standard way to do that is to write the solution set parametrically. So a parametric description is exactly a complete solution.

$$\boxed{\text{True}}$$

(v)

Whenever a linear system has free variables, the solution set contains a unique solution.

Answer: False.

Explanation: Free variables can take arbitrary values, so if a system has at least one free variable and is consistent, then it has infinitely many solutions, not a unique one.

$$\boxed{\text{False}}$$

(vi)

If one row in an echelon form of an augmented matrix is

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 5 \end{bmatrix},$$

the corresponding linear system is inconsistent.

Answer: True.

Explanation: This row represents the equation

$$0x_1 + 0x_2 + 0x_3 + 0x_4 = 5,$$

that is,

$$0 = 5,$$

which is impossible. Therefore the system has no solution.

$$\boxed{\text{True}}$$

End of solutions

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End of solutions