

MATH031 – SPRING 2026

Worksheet #5 Detailed Solutions

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Worksheet source: MATH031 – Spring 2026 Worksheet #5.

Exercise 1

Consider a linear system whose augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ -1 & 4 & 3 & 2 \\ 2 & -2 & a & 3 \end{array} \right].$$

For which values of a will the system have a unique solution? Explain.

A linear system has a unique solution when the coefficient matrix has a pivot in every variable column. Since there are 3 variables here, we need a pivot in each of the 3 columns of the coefficient matrix.

So we row reduce the coefficient part, or equivalently check when the coefficient matrix is invertible.

The coefficient matrix is

$$A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 4 & 3 \\ 2 & -2 & a \end{bmatrix}.$$

We row reduce.

Start with

$$\left[\begin{array}{ccc} 1 & 2 & 1 \\ -1 & 4 & 3 \\ 2 & -2 & a \end{array} \right].$$

Use the first row to eliminate the entries below it:

$$R_2 \leftarrow R_2 + R_1, \quad R_3 \leftarrow R_3 - 2R_1.$$

This gives

$$\left[\begin{array}{ccc} 1 & 2 & 1 \\ 0 & 6 & 4 \\ 0 & -6 & a - 2 \end{array} \right].$$

Now eliminate the entry below the pivot in column 2:

$$R_3 \leftarrow R_3 + R_2.$$

Then

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 6 & 4 \\ 0 & 0 & a+2 \end{bmatrix}.$$

For the system to have a unique solution, we need a pivot in the third column also. That happens exactly when

$$a + 2 \neq 0.$$

So the system has a unique solution when

$$\boxed{a \neq -2.}$$

Conclusion: The system has a unique solution for all real numbers a except

$$\boxed{a = -2.}$$

Exercise 2

Is

$$u = \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix}$$

in the subset of \mathbb{R}^3 spanned by the columns of

$$A = \begin{bmatrix} 5 & 8 & 7 \\ 0 & 1 & -1 \\ 1 & 3 & 0 \end{bmatrix}?$$

Why or why not?

A vector u is in the span of the columns of A exactly when the equation

$$Ax = u$$

has a solution.

So we solve

$$\begin{bmatrix} 5 & 8 & 7 \\ 0 & 1 & -1 \\ 1 & 3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix}.$$

This corresponds to the augmented matrix

$$\left[\begin{array}{ccc|c} 5 & 8 & 7 & 2 \\ 0 & 1 & -1 & -3 \\ 1 & 3 & 0 & 2 \end{array} \right].$$

It is convenient to interchange rows 1 and 3 so that we start with a pivot of 1:

$$R_1 \leftrightarrow R_3.$$

Then

$$\left[\begin{array}{ccc|c} 1 & 3 & 0 & 2 \\ 0 & 1 & -1 & -3 \\ 5 & 8 & 7 & 2 \end{array} \right].$$

Now eliminate the entry below the first pivot:

$$R_3 \leftarrow R_3 - 5R_1.$$

This gives

$$\left[\begin{array}{ccc|c} 1 & 3 & 0 & 2 \\ 0 & 1 & -1 & -3 \\ 0 & -7 & 7 & -8 \end{array} \right].$$

Next eliminate the entry below the pivot in column 2:

$$R_3 \leftarrow R_3 + 7R_2.$$

Then

$$\left[\begin{array}{ccc|c} 1 & 3 & 0 & 2 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & -29 \end{array} \right].$$

The last row represents

$$0 = -29,$$

which is impossible. Therefore the system is inconsistent.

So there is no vector x such that $Ax = u$.

Hence

$$\boxed{u \text{ is not in the span of the columns of } A.}$$

Conclusion: No, u is not in the span, because the equation $Ax = u$ is inconsistent.

Exercise 3

Evaluate the matrix products, if possible.

(a)

$$\begin{bmatrix} 6 & 5 \\ -4 & -3 \\ 7 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

This product is defined because the first matrix is 3×2 and the second is 2×1 .

Compute each entry by taking row-dot-column products:

$$\text{Row 1: } 6(1) + 5(-3) = 6 - 15 = -9,$$

$$\text{Row 2: } -4(1) + (-3)(-3) = -4 + 9 = 5,$$

$$\text{Row 3: } 7(1) + 6(-3) = 7 - 18 = -11.$$

So

$$\begin{bmatrix} 6 & 5 \\ -4 & -3 \\ 7 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} -9 \\ 5 \\ -11 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 2 \\ 6 \\ -1 \end{bmatrix} [1 \quad -1]$$

This product is also defined. The first matrix is 3×1 and the second is 1×2 , so the result will be a 3×2 matrix.

Multiply each entry of the column vector by the row vector:

$$\begin{bmatrix} 2 \\ 6 \\ -1 \end{bmatrix} [1 \quad -1] = \begin{bmatrix} 2(1) & 2(-1) \\ 6(1) & 6(-1) \\ -1(1) & -1(-1) \end{bmatrix}.$$

Therefore

$$\begin{bmatrix} 2 \\ 6 \\ -1 \end{bmatrix} [1 \quad -1] = \begin{bmatrix} 2 & -2 \\ 6 & -6 \\ -1 & 1 \end{bmatrix}$$

(c)

$$\begin{bmatrix} 8 & 3 & 1 \\ 5 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

This product is defined because the first matrix is 2×3 and the second is 3×1 . The result will be a 2×1 vector.

Compute:

$$\text{Row 1: } 8(1) + 3(1) + 1(1) = 12,$$

$$\text{Row 2: } 5(1) + 1(1) + 2(1) = 8.$$

So

$$\begin{bmatrix} 8 & 3 & 1 \\ 5 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 12 \\ 8 \end{bmatrix}$$

Exercise 4

Are the columns of

$$A = \begin{bmatrix} 0 & -8 & 5 \\ 3 & -7 & 4 \\ -1 & 5 & -4 \\ 1 & -3 & 2 \end{bmatrix}$$

linearly independent? Justify your answer.

The columns of A are linearly independent if and only if the homogeneous equation

$$Ax = 0$$

has only the trivial solution.

So we row reduce the matrix A .

Start with

$$\begin{bmatrix} 0 & -8 & 5 \\ 3 & -7 & 4 \\ -1 & 5 & -4 \\ 1 & -3 & 2 \end{bmatrix}.$$

It is convenient to interchange row 1 and row 4:

$$R_1 \leftrightarrow R_4.$$

Then

$$\begin{bmatrix} 1 & -3 & 2 \\ 3 & -7 & 4 \\ -1 & 5 & -4 \\ 0 & -8 & 5 \end{bmatrix}.$$

Now eliminate entries below the first pivot:

$$R_2 \leftarrow R_2 - 3R_1, \quad R_3 \leftarrow R_3 + R_1.$$

This gives

$$\begin{bmatrix} 1 & -3 & 2 \\ 0 & 2 & -2 \\ 0 & 2 & -2 \\ 0 & -8 & 5 \end{bmatrix}.$$

Next eliminate below the pivot in column 2:

$$R_3 \leftarrow R_3 - R_2, \quad R_4 \leftarrow R_4 + 4R_2.$$

Then

$$\begin{bmatrix} 1 & -3 & 2 \\ 0 & 2 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & -3 \end{bmatrix}.$$

Reordering the nonzero rows gives an echelon form like

$$\begin{bmatrix} 1 & -3 & 2 \\ 0 & 2 & -2 \\ 0 & 0 & -3 \\ 0 & 0 & 0 \end{bmatrix}.$$

There is a pivot in every column of the matrix, so there are no free variables. Therefore the homogeneous system $Ax = 0$ has only the trivial solution.

Hence the columns are linearly independent.

The columns of A are linearly independent.

Exercise 5

True or false? Briefly justify your answers.

(i)

If the equation $Ax = b$ is inconsistent, then b is not in the set spanned by the columns of A .

Answer: True.

Explanation: The equation $Ax = b$ is consistent exactly when b can be written as a linear combination of the columns of A . So if the system is inconsistent, then b is not in the span of the columns of A .

True

(ii)

Every matrix equation $Ax = b$ corresponds to a vector equation with the same solution set.

Answer: True.

Explanation: If the columns of A are a_1, \dots, a_n , then the matrix equation

$$Ax = b$$

is equivalent to the vector equation

$$x_1a_1 + x_2a_2 + \cdots + x_na_n = b.$$

These two forms describe exactly the same system, so they have the same solution set.

True

(iii)

The equation $Ax = b$ is inconsistent if the augmented matrix $[A \ b]$ has a pivot in every row.

Answer: False.

Explanation: A pivot in every row does *not* mean inconsistency. Inconsistency occurs when there is a pivot in the *augmented column* that gives a row like

$$[0 \ 0 \ \cdots \ 0 \ | \ c] \quad \text{with } c \neq 0.$$

It is possible for $[A \ b]$ to have a pivot in every row and still be consistent.

False

(iv)

If A is an $m \times n$ matrix whose columns do not span \mathbb{R}^m , then the equation $Ax = b$ is inconsistent for some $b \in \mathbb{R}^m$.

Answer: True.

Explanation: If the columns of A do not span \mathbb{R}^m , then there are some vectors $b \in \mathbb{R}^m$ that cannot be written as linear combinations of the columns of A . For those vectors b , the equation $Ax = b$ has no solution.

True

(v)

The solution set of a linear system whose augmented matrix is $[a_1 \ a_2 \ a_3 \ b]$ is the same as the solution set of $Ax = b$, if $A = [a_1 \ a_2 \ a_3]$.

Answer: True.

Explanation: The augmented matrix $[a_1 \ a_2 \ a_3 \ b]$ represents the system

$$x_1a_1 + x_2a_2 + x_3a_3 = b,$$

which is exactly the same as

$$Ax = b$$

when

$$A = [a_1 \ a_2 \ a_3].$$

So they have the same solution set.

True

(vi)

The homogeneous equation $Ax = 0$ has the trivial solution if and only if the equation has at least one free variable.

Answer: False.

Explanation: Every homogeneous system $Ax = 0$ always has the trivial solution $x = 0$, whether or not there are free variables. Free variables determine whether there are *nontrivial* solutions, not whether the trivial solution exists.

False

(vii)

If x is a nontrivial solution of $Ax = 0$, then every entry in x is nonzero.

Answer: False.

Explanation: A nontrivial solution simply means $x \neq 0$, so at least one entry is nonzero. Some entries may still be zero.

For example,

$$x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

is nontrivial because it is not the zero vector, even though two entries are zero.

False

(viii)

The solution set of $Ax = b$ is the set of all vectors of the form $p + v_h$, where v_h is any solution of $Ax = 0$.

Answer: True.

Explanation: If p is one particular solution of $Ax = b$, then every solution of $Ax = b$ can be written as

$$x = p + v_h,$$

where v_h is a solution of the homogeneous equation $Ax = 0$. This is the standard description of the complete solution set.

True

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End of solutions