# MATH031 – SPRING 2026

## Worksheet #6 Detailed Solutions

**Prepared by Khoi Vo**

---

**Worksheet source:** MATH031 – Spring 2026 Worksheet #6.

---

## Exercise 1

**Consider a linear system whose augmented matrix is**

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 2 & 5 & 3 & 0 \\ -1 & 1 & \beta & 0 \end{array}\right].$$

**Is it possible for the system to be inconsistent? For what values of $\beta$ will the system have infinitely many solutions?**

### Step 1: Observe that the system is homogeneous

Since the entire right-hand side is zero, the system is of the form

$$Ax = 0.$$

A homogeneous system is *always consistent*, because the trivial solution

$$x = 0$$

always satisfies the system.

Therefore, it is **not possible** for this system to be inconsistent.

$$\boxed{\text{The system can never be inconsistent.}}$$

### Step 2: Determine when the system has infinitely many solutions

A homogeneous system has infinitely many solutions exactly when there is at least one free variable. Since there are 3 variables here, that happens when the coefficient matrix does *not* have a pivot in every column.

The coefficient matrix is

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 3 \\ -1 & 1 & \beta \end{bmatrix}.$$

We row reduce it.

Start with

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 3 \\ -1 & 1 & \beta \end{bmatrix}.$$

Use the first row to eliminate entries below the first pivot:

$$R_2 \leftarrow R_2 - 2R_1, \qquad R_3 \leftarrow R_3 + R_1.$$

This gives

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 3 & \beta + 1 \end{bmatrix}.$$

Now eliminate the entry below the pivot in column 2:

$$R_3 \leftarrow R_3 - 3R_2.$$

Then

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & \beta - 2 \end{bmatrix}.$$

For the system to have infinitely many solutions, we need *no pivot* in the third column, so we require

$$\beta - 2 = 0.$$

Hence

$$\boxed{\beta = 2.}$$

## Conclusion

$$\boxed{\text{The system is never inconsistent.}}$$

$$\boxed{\text{The system has infinitely many solutions when } \beta = 2.}$$

# Exercise 2

**Find the value of $h$ for which the vectors**

$$\begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}, \qquad \begin{bmatrix} 3 \\ -5 \\ 7 \end{bmatrix}, \qquad \begin{bmatrix} -1 \\ 5 \\ h \end{bmatrix}$$

**are linearly dependent.**

Three vectors in $\mathbb{R}^3$ are linearly dependent exactly when the matrix having these vectors as columns does *not* have a pivot in every column. Equivalently, its determinant is zero, or its rows reduce to fewer than three pivots.

Let

$$A = \begin{bmatrix} 1 & 3 & -1 \\ -1 & -5 & 5 \\ 4 & 7 & h \end{bmatrix}.$$

We row reduce.

Start with

$$\begin{bmatrix} 1 & 3 & -1 \\ -1 & -5 & 5 \\ 4 & 7 & h \end{bmatrix}.$$

Use the first row to eliminate below the pivot:

$$R_2 \leftarrow R_2 + R_1, \qquad R_3 \leftarrow R_3 - 4R_1.$$

This gives

$$\begin{bmatrix} 1 & 3 & -1 \\ 0 & -2 & 4 \\ 0 & -5 & h+4 \end{bmatrix}.$$

Now eliminate the entry below the pivot in column 2. One convenient way is

$$R_3 \leftarrow 2R_3 - 5R_2.$$

Then

$$2R_3 = \begin{bmatrix} 0 & -10 & 2h+8 \end{bmatrix}, \qquad 5R_2 = \begin{bmatrix} 0 & -10 & 20 \end{bmatrix},$$

so

$$R_3 = \begin{bmatrix} 0 & 0 & 2h-12 \end{bmatrix}.$$

Thus the matrix becomes

$$\begin{bmatrix} 1 & 3 & -1 \\ 0 & -2 & 4 \\ 0 & 0 & 2h-12 \end{bmatrix}.$$

For the vectors to be linearly dependent, the third pivot must disappear. Therefore we need

$$2h - 12 = 0.$$

So

$$h = 6.$$

Therefore,

$$\boxed{h = 6.}$$

# Exercise 3

**Determine by inspection whether the given vectors are linearly independent or not. Briefly explain.**

## (a)

$$\begin{bmatrix} 5 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 8 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} -1 \\ 7 \end{bmatrix}$$

These are 4 vectors in $\mathbb{R}^2$. In $\mathbb{R}^2$, any set with more than 2 vectors must be linearly dependent, because there cannot be more than 2 pivot positions.

Therefore,

$$\boxed{\text{The vectors are linearly dependent.}}$$

## (b)

$$\begin{bmatrix} 3 \\ 5 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -6 \\ 5 \\ 4 \end{bmatrix}$$

Any set containing the zero vector is automatically linearly dependent, because the zero vector can be written as a trivial multiple of the other vectors, and it prevents the set from being independent.

Therefore,

$$\boxed{\text{The vectors are linearly dependent.}}$$

## (c)

$$\begin{bmatrix} -8 \\ 12 \\ -4 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix}$$

Two vectors are linearly dependent if one is a scalar multiple of the other.

Check whether

$$\begin{bmatrix} -8 \\ 12 \\ -4 \end{bmatrix} = c \begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix}$$

for some scalar $c$.

From the first entry, $c = -4$. From the second entry, $c = -4$ also works since $-4(-3) = 12$. But from the third entry,

$$-4(-1) = 4 \neq -4.$$

So the vectors are *not* scalar multiples of each other.

Since there are only two vectors, and neither is a scalar multiple of the other, they are linearly independent.

Therefore,

$$\boxed{\text{The vectors are linearly independent.}}$$

# Exercise 4

**Decide whether the following are linear transformations.**

Recall that a map $T$ is a linear transformation if for all vectors $u, v$ and scalars $c$,

$$T(u+v) = T(u) + T(v) \qquad \text{and} \qquad T(cu) = cT(u).$$

In practice, formulas made only from linear combinations of the input variables are linear, while products like $xy$, $xyz$, squares, constants, etc. usually destroy linearity.

## (a)

$$T : \mathbb{R}^2 \to \mathbb{R}^3, \qquad T(x, y) = (x + 2y,\ -x + y,\ 0).$$

Each component is a linear combination of $x$ and $y$, and there is no constant term or product of variables.

We can verify directly:

$$T(x_1 + x_2,\ y_1 + y_2) = ((x_1 + x_2) + 2(y_1 + y_2),\ -(x_1 + x_2) + (y_1 + y_2),\ 0).$$

This simplifies to

$$(x_1 + 2y_1,\ -x_1 + y_1,\ 0) + (x_2 + 2y_2,\ -x_2 + y_2,\ 0) = T(x_1, y_1) + T(x_2, y_2).$$

Also,

$$T(cx, cy) = (cx + 2cy,\ -cx + cy,\ 0) = c(x + 2y,\ -x + y,\ 0) = cT(x, y).$$

So $T$ is linear.

$$\boxed{T \text{ is a linear transformation.}}$$

## (b)

$$S : \mathbb{R}^3 \to \mathbb{R}^2, \qquad S(x, y, z) = (x + y + z,\ xyz).$$

The first component $x + y + z$ is linear, but the second component $xyz$ is not linear because it involves multiplication of variables.

A quick test is to check whether $S(0) = 0$ and whether additivity holds.

We do have

$$S(0, 0, 0) = (0, 0),$$

but additivity fails. For example,

$$S(1,0,1) = (2,0), \qquad S(0,1,1) = (2,0).$$

So

$$S(1,0,1) + S(0,1,1) = (4,0).$$

But

$$S((1,0,1) + (0,1,1)) = S(1,1,2) = (4,2).$$

These are not equal, so $S$ is not linear.

Therefore,

$$\boxed{S \text{ is not a linear transformation.}}$$

## (c)

$$R : \mathbb{R}^2 \to \mathbb{R}^2, \qquad R(u,v) = (u,\ u + 4v).$$

Both components are linear combinations of $u$ and $v$.

Check additivity:

$$R(u_1 + u_2,\ v_1 + v_2) = (u_1 + u_2,\ (u_1 + u_2) + 4(v_1 + v_2)).$$

This equals

$$(u_1,\ u_1 + 4v_1) + (u_2,\ u_2 + 4v_2) = R(u_1, v_1) + R(u_2, v_2).$$

Check scalar multiplication:

$$R(cu, cv) = (cu,\ cu + 4cv) = c(u,\ u + 4v) = cR(u,v).$$

So $R$ is linear.

$$\boxed{R \text{ is a linear transformation.}}$$

# Exercise 5

**True or false? Briefly justify your answers.**

## (i)

**The columns of a matrix $A$ are linearly independent if the equation $Ax = 0$ has the trivial solution.**

**Answer: False.**

**Explanation:** Every homogeneous system $Ax = 0$ always has the trivial solution $x = 0$. That alone is not enough to guarantee linear independence. The columns are linearly independent only if $Ax = 0$ has *only* the trivial solution.

$$\boxed{\text{False}}$$

## (ii)

**Two vectors are linearly dependent if and only if they lie on a line through the origin.**

**Answer: True.**

**Explanation:** Two vectors are linearly dependent exactly when one is a scalar multiple of the other. Geometrically, that means they lie on the same line through the origin.

$$\boxed{\text{True}}$$

## (iii)

**If $S$ is a linearly dependent set, then each vector is a linear combination of the other vectors in $S$.**

**Answer: False.**

**Explanation:** If a set is linearly dependent, then *at least one* vector in the set can be written as a linear combination of the others. It does not mean that *every* vector has this property.

$$\boxed{\text{False}}$$

## (iv)

**If a set contains fewer vectors than there are entries in the vectors, then the set is linearly independent.**

**Answer: False.**

**Explanation:** Having fewer vectors than entries does not guarantee independence. For example, in $\mathbb{R}^3$, the two vectors

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

are linearly dependent, even though there are only 2 vectors in $\mathbb{R}^3$.

$$\boxed{\text{False}}$$

## (v)

**If a set in $\mathbb{R}^n$ is linearly dependent, then the set contains more vectors than there are entries in each vector.**

**Answer: False.**

**Explanation:** A linearly dependent set does *not* have to contain more than $n$ vectors. Dependence can happen even with fewer than $n$ vectors, for example if one vector is a scalar multiple of another.

Example in $\mathbb{R}^3$:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

is linearly dependent, but it contains only 2 vectors.

$$\boxed{\text{False}}$$

---

# Copyright Notice

**End of solutions**