MATH031 – SPRING 2026
Worksheet #7 Detailed Solutions

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Exercise 1

Find the standard matrix of the linear transformation T in the following cases.

Recall that if

$$T : \mathbb{R}^n \to \mathbb{R}^m$$

is linear, then its standard matrix is

$$A = \begin{bmatrix} T(e_1) & T(e_2) & \cdots & T(e_n) \end{bmatrix},$$

where $e_1, \dots, e_n$ are the standard basis vectors in $\mathbb{R}^n$.

(a)

$$T : \mathbb{R}^2 \to \mathbb{R}^4, \qquad T(1,0) = (2,1,2,1), \qquad T(0,1) = (-5,2,0,0).$$

The vectors $(1,0)$ and $(0,1)$ are exactly the standard basis vectors of $\mathbb{R}^2$, namely

$$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Therefore the standard matrix has columns $T(e_1)$ and $T(e_2)$:

$$A = \begin{bmatrix} 2 & -5 \\ 1 & 2 \\ 2 & 0 \\ 1 & 0 \end{bmatrix}.$$

So the standard matrix is

$$\boxed{\begin{bmatrix} 2 & -5 \\ 1 & 2 \\ 2 & 0 \\ 1 & 0 \end{bmatrix}}$$

(b)

$$T : \mathbb{R}^3 \to \mathbb{R}^2$$

with

$$T(1,0,0) = (1,3), \qquad T(0,1,0) = (4,2), \qquad T(0,0,1) = (-5,4).$$

These are the images of the standard basis vectors

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Thus the standard matrix is

$$A = \begin{bmatrix} 1 & 4 & -5 \\ 3 & 2 & 4 \end{bmatrix}.$$

Therefore,

$$\boxed{\begin{bmatrix} 1 & 4 & -5 \\ 3 & 2 & 4 \end{bmatrix}}$$

(c)

$$T : \mathbb{R}^2 \to \mathbb{R}^2$$

rotates points about $(0,0)$ by

$$\frac{3\pi}{2}$$

radians counterclockwise.

Recall the standard rotation matrix for angle θ:

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Here $\theta = \frac{3\pi}{2}$, so

$$\cos\left(\frac{3\pi}{2}\right) = 0, \qquad \sin\left(\frac{3\pi}{2}\right) = -1.$$

Substituting into the rotation matrix gives

$$A = \begin{bmatrix} 0 & -(-1) \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

So the standard matrix is

$$\boxed{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}$$

Exercise 2

If $T : \mathbb{R}^3 \to \mathbb{R}^4$ has $T(u) = (1, 0, 2, 1)$ and $T(v) = (0, -1, 3, 3)$, find $T(4u)$, $T(-2v)$, and $T(3u + 5v)$.

Because T is a linear transformation, it satisfies

$$T(cw) = cT(w) \qquad \text{and} \qquad T(w_1 + w_2) = T(w_1) + T(w_2).$$

(a) Find $T(4u)$

Using homogeneity,

$$T(4u) = 4T(u) = 4(1, 0, 2, 1) = (4, 0, 8, 4).$$

Therefore,

$$\boxed{T(4u) = (4, 0, 8, 4).}$$

(b) Find $T(-2v)$

Again using homogeneity,

$$T(-2v) = -2T(v) = -2(0, -1, 3, 3) = (0, 2, -6, -6).$$

Therefore,

$$\boxed{T(-2v) = (0, 2, -6, -6).}$$

(c) Find $T(3u + 5v)$

Using linearity,

$$T(3u + 5v) = 3T(u) + 5T(v).$$

Now compute:

$$3T(u) = 3(1, 0, 2, 1) = (3, 0, 6, 3),$$
$$5T(v) = 5(0, -1, 3, 3) = (0, -5, 15, 15).$$

Add the vectors:

$$(3, 0, 6, 3) + (0, -5, 15, 15) = (3, -5, 21, 18).$$

Thus,

$$\boxed{T(3u + 5v) = (3, -5, 21, 18).}$$

Exercise 3

Given

$$A = \begin{bmatrix} 3 & 1 & 4 \\ -2 & 0 & 1 \\ 1 & 2 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 & 2 \\ -3 & 1 & 1 \\ 2 & -4 & 1 \end{bmatrix},$$

compute:

$$\text{(i) } 2A - 3B, \qquad \text{(ii) } -3A + 4B, \qquad \text{(iii) } AB, \qquad \text{(iv) } BA.$$

(i) Compute $2A - 3B$

First compute

$$2A = \begin{bmatrix} 6 & 2 & 8 \\ -4 & 0 & 2 \\ 2 & 4 & 4 \end{bmatrix}, \qquad 3B = \begin{bmatrix} 3 & 0 & 6 \\ -9 & 3 & 3 \\ 6 & -12 & 3 \end{bmatrix}.$$

Now subtract:

$$2A - 3B = \begin{bmatrix} 6-3 & 2-0 & 8-6 \\ -4-(-9) & 0-3 & 2-3 \\ 2-6 & 4-(-12) & 4-3 \end{bmatrix}.$$

So

$$\boxed{2A - 3B = \begin{bmatrix} 3 & 2 & 2 \\ 5 & -3 & -1 \\ -4 & 16 & 1 \end{bmatrix}}$$

(ii) Compute $-3A + 4B$

First compute

$$-3A = \begin{bmatrix} -9 & -3 & -12 \\ 6 & 0 & -3 \\ -3 & -6 & -6 \end{bmatrix}, \qquad 4B = \begin{bmatrix} 4 & 0 & 8 \\ -12 & 4 & 4 \\ 8 & -16 & 4 \end{bmatrix}.$$

Add them:

$$-3A + 4B = \begin{bmatrix} -9+4 & -3+0 & -12+8 \\ 6+(-12) & 0+4 & -3+4 \\ -3+8 & -6+(-16) & -6+4 \end{bmatrix}.$$

Thus

$$\boxed{-3A + 4B = \begin{bmatrix} -5 & -3 & -4 \\ -6 & 4 & 1 \\ 5 & -22 & -2 \end{bmatrix}}$$

(iii) Compute AB

We multiply rows of A by columns of B.

The columns of B are

$$\begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, \qquad \begin{bmatrix} 0 \\ 1 \\ -4 \end{bmatrix}, \qquad \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}.$$

The first row of A is $(3, 1, 4)$:

$$(3, 1, 4) \cdot (1, -3, 2) = 3 - 3 + 8 = 8,$$

$$(3, 1, 4) \cdot (0, 1, -4) = 0 + 1 - 16 = -15,$$
$$(3, 1, 4) \cdot (2, 1, 1) = 6 + 1 + 4 = 11.$$

The second row of A is $(-2, 0, 1)$:

$$(-2, 0, 1) \cdot (1, -3, 2) = -2 + 0 + 2 = 0,$$

$$(-2, 0, 1) \cdot (0, 1, -4) = 0 + 0 - 4 = -4,$$
$$(-2, 0, 1) \cdot (2, 1, 1) = -4 + 0 + 1 = -3.$$

The third row of A is $(1, 2, 2)$:

$$(1, 2, 2) \cdot (1, -3, 2) = 1 - 6 + 4 = -1,$$

$$(1, 2, 2) \cdot (0, 1, -4) = 0 + 2 - 8 = -6,$$
$$(1, 2, 2) \cdot (2, 1, 1) = 2 + 2 + 2 = 6.$$

Therefore,

$$\boxed{AB = \begin{bmatrix} 8 & -15 & 11 \\ 0 & -4 & -3 \\ -1 & -6 & 6 \end{bmatrix}}$$

(iv) Compute BA

Now multiply rows of B by columns of A.

The columns of A are

$$\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}, \qquad \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \qquad \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}.$$

The first row of B is $(1,0,2)$:

$$(1,0,2)\cdot(3,-2,1) = 3+0+2 = 5,$$

$$(1,0,2)\cdot(1,0,2) = 1+0+4 = 5,$$

$$(1,0,2)\cdot(4,1,2) = 4+0+4 = 8.$$

The second row of B is $(-3,1,1)$:

$$(-3,1,1)\cdot(3,-2,1) = -9-2+1 = -10,$$

$$(-3,1,1)\cdot(1,0,2) = -3+0+2 = -1,$$

$$(-3,1,1)\cdot(4,1,2) = -12+1+2 = -9.$$

The third row of B is $(2,-4,1)$:

$$(2,-4,1)\cdot(3,-2,1) = 6+8+1 = 15,$$

$$(2,-4,1)\cdot(1,0,2) = 2+0+2 = 4,$$

$$(2,-4,1)\cdot(4,1,2) = 8-4+2 = 6.$$

Hence

$$\boxed{BA = \begin{bmatrix} 5 & 5 & 8 \\ -10 & -1 & -9 \\ 15 & 4 & 6 \end{bmatrix}}$$

Exercise 4

Assume that A is a 3×2 matrix, B is 3×3, and C is 3×4. Among the nine products

$$A^2,\ AB,\ AC,\ BA,\ B^2,\ BC,\ CA,\ CB,\ C^2,$$

decide which ones are well-defined and which ones are not. What are the sizes of the products which are well-defined?

Recall that a product

$$XY$$

is defined only when the number of columns of X equals the number of rows of Y.

We are given:

$$A:\ 3\times 2, \qquad B:\ 3\times 3, \qquad C:\ 3\times 4.$$

Now check each product.

$A^2 = AA$

$$(3\times 2)(3\times 2)$$

is **not defined**, because the inner dimensions 2 and 3 do not match.

$$\boxed{A^2 \text{ is not defined.}}$$

AB

$$(3\times 2)(3\times 3)$$

is **not defined**, because $2 \neq 3$.

$$\boxed{AB \text{ is not defined.}}$$

AC

$$(3\times 2)(3\times 4)$$

is **not defined**, because $2 \neq 3$.

$$\boxed{AC \text{ is not defined.}}$$

BA

$$(3 \times 3)(3 \times 2)$$

is defined, and the result has size

$$3 \times 2.$$

$$\boxed{BA \text{ is defined and has size } 3 \times 2.}$$

$B^2 = BB$

$$(3 \times 3)(3 \times 3)$$

is defined, and the result has size

$$3 \times 3.$$

$$\boxed{B^2 \text{ is defined and has size } 3 \times 3.}$$

BC

$$(3 \times 3)(3 \times 4)$$

is defined, and the result has size

$$3 \times 4.$$

$$\boxed{BC \text{ is defined and has size } 3 \times 4.}$$

CA

$$(3 \times 4)(3 \times 2)$$

is **not defined**, because $4 \neq 3$.

$$\boxed{CA \text{ is not defined.}}$$

CB

$$(3 \times 4)(3 \times 3)$$

is **not defined**, because $4 \neq 3$.

$$\boxed{CB \text{ is not defined.}}$$

$C^2 = CC$

$$(3 \times 4)(3 \times 4)$$

is **not defined**, because $4 \neq 3$.

$$\boxed{C^2 \text{ is not defined.}}$$

Final summary

The only products that are well-defined are:

$$\boxed{BA \text{ of size } 3 \times 2, \qquad B^2 \text{ of size } 3 \times 3, \qquad BC \text{ of size } 3 \times 4.}$$

Exercise 5

True or false? Discuss and briefly justify your answers.

(i)

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is completely determined by its effect on the columns of the $n \times n$ identity matrix.

Answer: True.

Explanation: The columns of the $n \times n$ identity matrix are the standard basis vectors

$$e_1, \ldots, e_n.$$

Every vector in $\mathbb{R}^n$ can be written as a linear combination of these basis vectors, and linearity tells us how T acts on any such combination. Therefore knowing $T(e_1), \ldots, T(e_n)$ completely determines T.

True

(ii)

A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one if each vector in $\mathbb{R}^n$ maps onto a unique vector in $\mathbb{R}^m$.

Answer: False.

Explanation: Every function assigns exactly one output to each input, so this statement describes what it means to be a function, not what it means to be one-to-one. A map is one-to-one when *different inputs produce different outputs.* Equivalently,

$$T(x_1) = T(x_2) \implies x_1 = x_2.$$

False

(iii)

The columns of the standard matrix for a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ are the images of the columns of the $n \times n$ identity matrix.

Answer: True.

Explanation: The standard matrix of T is

$$A = \begin{bmatrix} T(e_1) & T(e_2) & \cdots & T(e_n) \end{bmatrix},$$

and the columns of the identity matrix are exactly $e_1, \ldots, e_n$.

True

(iv)

When two linear transformations are performed one after another, the combined effect may not always be a linear transformation.

Answer: False.

Explanation: The composition of two linear transformations is always linear, provided the dimensions match so that the composition makes sense.

False

(v)

If A and B are 2×2 with columns a_1, a_2 and b_1, b_2, respectively, then we have

$$AB = [a_1 b_1 \;\; a_2 b_2].$$

Answer: False.

Explanation: Matrix multiplication is not done by multiplying columns entry-by-entry like this. The correct column description is:

$$AB = \begin{bmatrix} Ab_1 & Ab_2 \end{bmatrix}.$$

False

(vi)

If A and B are 3×3 and $B = [b_1 \;\; b_2 \;\; b_3]$, then

$$AB = [Ab_1 + Ab_2 + Ab_3].$$

Answer: False.

Explanation: The correct formula is

$$AB = \begin{bmatrix} Ab_1 & Ab_2 & Ab_3 \end{bmatrix}.$$

The statement given incorrectly adds the columns together.

False

(vii)

Each column of AB is a linear combination of the columns of B, using weights from the corresponding column of A.

Answer: False.

Explanation: Each column of AB is a linear combination of the columns of A, using weights from the corresponding column of B. In fact, if b_j is the jth column of B, then the jth column of AB is

$$Ab_j,$$

which is a linear combination of the columns of A.

False

(viii)

The second row of AB is the second row of A multiplied on the right by B.

Answer: True.

Explanation: When computing AB, each row of the product is obtained by multiplying the corresponding row of A by B. Therefore the second row of AB is exactly the second row of A multiplied on the right by B.

True

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End of solutions