
MATH031 – SPRING 2026

Worksheet #4 Detailed Solutions

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Exercise 1

Write vector equations equivalent to the given systems.

The idea is to rewrite each system in the form

$$x_1v_1 + x_2v_2 + x_3v_3 = b,$$

where the vectors v_1, v_2, v_3 come from the columns of the coefficient matrix, and b is the right-hand side vector.

(a)

The system is

$$\begin{cases} x_2 + 5x_3 = 0, \\ 4x_1 + 6x_2 - x_3 = 0, \\ -x_1 + 3x_2 - 8x_3 = 0. \end{cases}$$

First, write the coefficient of each variable column by column.

For x_1 , the coefficients are

$$\begin{bmatrix} 0 \\ 4 \\ -1 \end{bmatrix}.$$

For x_2 , the coefficients are

$$\begin{bmatrix} 1 \\ 6 \\ 3 \end{bmatrix}.$$

For x_3 , the coefficients are

$$\begin{bmatrix} 5 \\ -1 \\ -8 \end{bmatrix}.$$

The right-hand side is

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

So the equivalent vector equation is

$$x_1 \begin{bmatrix} 0 \\ 4 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 6 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} 5 \\ -1 \\ -8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Thus,

$$\boxed{x_1 \begin{bmatrix} 0 \\ 4 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 6 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} 5 \\ -1 \\ -8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}$$

So here,

$$v_1 = \begin{bmatrix} 0 \\ 4 \\ -1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 6 \\ 3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 5 \\ -1 \\ -8 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

(b)

The system is

$$\begin{cases} 4x_1 + x_2 + 3x_3 = 9, \\ x_1 - 7x_2 - 2x_3 = 2, \\ 8x_1 + 6x_2 - 5x_3 = 15. \end{cases}$$

Again, collect the coefficients of each variable by columns.

For x_1 , the coefficients are

$$\begin{bmatrix} 4 \\ 1 \\ 8 \end{bmatrix}.$$

For x_2 , the coefficients are

$$\begin{bmatrix} 1 \\ -7 \\ 6 \end{bmatrix}.$$

For x_3 , the coefficients are

$$\begin{bmatrix} 3 \\ -2 \\ -5 \end{bmatrix}.$$

The right-hand side vector is

$$\begin{bmatrix} 9 \\ 2 \\ 15 \end{bmatrix}.$$

Therefore the vector equation is

$$x_1 \begin{bmatrix} 4 \\ 1 \\ 8 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ -7 \\ 6 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ -2 \\ -5 \end{bmatrix} = \begin{bmatrix} 9 \\ 2 \\ 15 \end{bmatrix}.$$

Hence,

$$\boxed{x_1 \begin{bmatrix} 4 \\ 1 \\ 8 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ -7 \\ 6 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ -2 \\ -5 \end{bmatrix} = \begin{bmatrix} 9 \\ 2 \\ 15 \end{bmatrix}}$$

So here,

$$v_1 = \begin{bmatrix} 4 \\ 1 \\ 8 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ -7 \\ 6 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 3 \\ -2 \\ -5 \end{bmatrix}, \quad b = \begin{bmatrix} 9 \\ 2 \\ 15 \end{bmatrix}.$$

Exercise 2

In each case, determine whether b is a linear combination of a_1, a_2, a_3 .

To determine whether b is a linear combination of a_1, a_2, a_3 , we ask whether there exist scalars x_1, x_2, x_3 such that

$$x_1 a_1 + x_2 a_2 + x_3 a_3 = b.$$

This is equivalent to solving a linear system.

(a)

Given

$$a_1 = (1, -2, 0), \quad a_2 = (0, 1, 2), \quad a_3 = (5, -6, 8), \quad b = (2, -1, 6).$$

We want to solve

$$x_1(1, -2, 0) + x_2(0, 1, 2) + x_3(5, -6, 8) = (2, -1, 6).$$

Matching components gives the system

$$\begin{cases} x_1 + 5x_3 = 2, \\ -2x_1 + x_2 - 6x_3 = -1, \\ 2x_2 + 8x_3 = 6. \end{cases}$$

Write the augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ -2 & 1 & -6 & -1 \\ 0 & 2 & 8 & 6 \end{array} \right].$$

Now row reduce.

First,

$$R_2 \leftarrow R_2 + 2R_1$$

gives

$$\left[\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ 0 & 1 & 4 & 3 \\ 0 & 2 & 8 & 6 \end{array} \right].$$

Next,

$$R_3 \leftarrow R_3 - 2R_2$$

gives

$$\left[\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{array} \right].$$

This system is consistent, so a solution exists. Therefore b is a linear combination of a_1, a_2, a_3 .

Let $x_3 = t$. Then from row 2,

$$x_2 + 4t = 3 \implies x_2 = 3 - 4t,$$

and from row 1,

$$x_1 + 5t = 2 \implies x_1 = 2 - 5t.$$

So there are infinitely many choices of coefficients. One simple choice is to take $t = 0$, giving

$$x_1 = 2, \quad x_2 = 3, \quad x_3 = 0.$$

Thus

$$b = 2a_1 + 3a_2 + 0a_3.$$

Therefore,

Yes, b is a linear combination of a_1, a_2, a_3 .

(b)

Given

$$a_1 = (1, -2, 2), \quad a_2 = (0, 5, 5), \quad a_3 = (2, 0, 8), \quad b = (-5, 11, -7).$$

We want to solve

$$x_1(1, -2, 2) + x_2(0, 5, 5) + x_3(2, 0, 8) = (-5, 11, -7).$$

Matching components gives

$$\begin{cases} x_1 + 2x_3 = -5, \\ -2x_1 + 5x_2 = 11, \\ 2x_1 + 5x_2 + 8x_3 = -7. \end{cases}$$

The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & -5 \\ -2 & 5 & 0 & 11 \\ 2 & 5 & 8 & -7 \end{array} \right].$$

Now row reduce.

First,

$$R_2 \leftarrow R_2 + 2R_1, \quad R_3 \leftarrow R_3 - 2R_1.$$

Then

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & -5 \\ 0 & 5 & 4 & 1 \\ 0 & 5 & 4 & 3 \end{array} \right].$$

Next,

$$R_3 \leftarrow R_3 - R_2$$

gives

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & -5 \\ 0 & 5 & 4 & 1 \\ 0 & 0 & 0 & 2 \end{array} \right].$$

The last row represents

$$0 = 2,$$

which is impossible. So the system is inconsistent.

Therefore,

No, b is not a linear combination of a_1, a_2, a_3 .

Exercise 3

Let $v_1 = (1, 0, -2)$ and $v_2 = (-3, 1, 8)$, and $y = (h, -5, -3)$. For which values of h is y in the plane spanned by v_1 and v_2 ?

We want to know when

$$y = c_1v_1 + c_2v_2$$

for some scalars c_1, c_2 .

So solve

$$c_1(1, 0, -2) + c_2(-3, 1, 8) = (h, -5, -3).$$

Matching coordinates gives

$$\begin{cases} c_1 - 3c_2 = h, \\ c_2 = -5, \\ -2c_1 + 8c_2 = -3. \end{cases}$$

From the second equation,

$$c_2 = -5.$$

Substitute into the third equation:

$$-2c_1 + 8(-5) = -3$$

$$-2c_1 - 40 = -3$$

$$-2c_1 = 37$$

$$c_1 = -\frac{37}{2}.$$

Now substitute c_1 and c_2 into the first equation:

$$h = c_1 - 3c_2$$

$$h = -\frac{37}{2} - 3(-5)$$

$$h = -\frac{37}{2} + 15$$

$$h = -\frac{37}{2} + \frac{30}{2}$$

$$h = -\frac{7}{2}.$$

Therefore,

$$\boxed{h = -\frac{7}{2}}.$$

So y lies in the plane spanned by v_1 and v_2 exactly when

$$h = -\frac{7}{2}.$$

Exercise 4

True or false? Briefly justify your answers.

(i)

In \mathbb{R}^n , $4v_1$ is a linear combination of v_1, v_2, v_3 .

Answer: True.

Explanation: A linear combination of v_1, v_2, v_3 has the form

$$c_1v_1 + c_2v_2 + c_3v_3.$$

If we choose

$$c_1 = 4, \quad c_2 = 0, \quad c_3 = 0,$$

then

$$4v_1 = 4v_1 + 0v_2 + 0v_3.$$

So $4v_1$ is indeed a linear combination of the three vectors.

True

(ii)

The scalar weights in a linear combination $c_1v_1 + \cdots + c_pv_p$ can't be all 0.

Answer: False.

Explanation: It is allowed for all the scalars to be zero. In that case,

$$0v_1 + 0v_2 + \cdots + 0v_p = 0.$$

This is still a valid linear combination, called the trivial linear combination.

False

(iii)

Given ten vectors v_1, \dots, v_{10} in \mathbb{R}^2 , it is always possible to write any vector b as a linear combination of v_1, \dots, v_{10} .

Answer: False.

Explanation: Just having ten vectors in \mathbb{R}^2 does not guarantee that they span all of \mathbb{R}^2 . For example, if all ten vectors are the zero vector, or if they all lie on the same line, then many vectors in \mathbb{R}^2 cannot be written as linear combinations of them.

So this is not always true.

False

(iv)

When u and v are nonzero vectors, $\text{Span}\{u, v\}$ contains the line through u and the origin 0 .

Answer: True.

Explanation: The line through u and the origin consists of all scalar multiples of u , that is,

$$\{cu : c \in \mathbb{R}\}.$$

But every vector of the form cu can be written as

$$cu + 0v,$$

which is a linear combination of u and v . Therefore every point on that line lies in $\text{Span}\{u, v\}$.

True

(v)

Asking whether the linear system whose augmented matrix is

$$\begin{bmatrix} a_1 & a_2 & a_3 & b \end{bmatrix}$$

has a solution amounts to asking whether b is in $\text{Span}\{a_1, a_2, a_3\}$.

Answer: True.

Explanation: The system

$$x_1a_1 + x_2a_2 + x_3a_3 = b$$

has a solution exactly when there exist scalars x_1, x_2, x_3 making this equation true. But that is exactly the definition of saying that b is a linear combination of a_1, a_2, a_3 , or equivalently,

$$b \in \text{Span}\{a_1, a_2, a_3\}.$$

True

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End of solutions