

MATH 031 Applied Linear Algebra

Quiz #3 Detailed Solutions

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Question 1

Explain why the following sets are subspaces of \mathbb{R}^5 .

Question 1(a)

Consider the set of all vectors of the form

$$(s + t, s - t, s + 2t, s, t),$$

where $s, t \in \mathbb{R}$.

We want to show that this set is a subspace of \mathbb{R}^5 .

Rewrite the vector by separating the terms involving s and the terms involving t :

$$(s + t, s - t, s + 2t, s, t) = (s, s, s, s, 0) + (t, -t, 2t, 0, t).$$

Now factor out s and t :

$$(s + t, s - t, s + 2t, s, t) = s(1, 1, 1, 1, 0) + t(1, -1, 2, 0, 1).$$

Therefore, the set can be written as

$$\{s(1, 1, 1, 1, 0) + t(1, -1, 2, 0, 1) : s, t \in \mathbb{R}\}.$$

This means the set is

$$\text{span}\{(1, 1, 1, 1, 0), (1, -1, 2, 0, 1)\}.$$

Since the span of any collection of vectors in \mathbb{R}^5 is always a subspace of \mathbb{R}^5 , the given set is a subspace of \mathbb{R}^5 .

The set is a subspace because it is the span of two vectors in \mathbb{R}^5 .

Question 1(b)

Consider the set of all vectors

$$(x, y, z, w, t) \in \mathbb{R}^5$$

such that

$$x + 2y + 3z + 4y + 5t = 0$$

and

$$x - y + z - w + t = 0.$$

Notice that the first equation has both $2y$ and $4y$. Combining like terms, we get

$$x + 6y + 3z + 5t = 0.$$

So the system becomes

$$\begin{cases} x + 6y + 3z + 0w + 5t = 0, \\ x - y + z - w + t = 0. \end{cases}$$

This is a homogeneous system of linear equations. In matrix form, we can write it as

$$\begin{bmatrix} 1 & 6 & 3 & 0 & 5 \\ 1 & -1 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Therefore, the given set is the nullspace of the matrix

$$B = \begin{bmatrix} 1 & 6 & 3 & 0 & 5 \\ 1 & -1 & 1 & -1 & 1 \end{bmatrix}.$$

That is,

$$S = \text{Null}(B).$$

The nullspace of any matrix is always a subspace because it is the solution set of a homogeneous linear system.

The set is a subspace because it is the nullspace of a matrix.

Question 2

Find an explicit description of the nullspace of

$$A = \begin{bmatrix} 1 & 5 & -4 & -3 & 1 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

We want to solve

$$A\mathbf{x} = \mathbf{0},$$

where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}.$$

Since we are solving a homogeneous system, we row-reduce the matrix A .

$$A = \begin{bmatrix} 1 & 5 & -4 & -3 & 1 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The pivot in the second row is already a 1. To get reduced echelon form, we eliminate the entry above this pivot.

Use the row operation

$$R_1 \leftarrow R_1 - 5R_2.$$

Then

$$R_1 = [1 \ 5 \ -4 \ -3 \ 1] - 5[0 \ 1 \ -2 \ 1 \ 0].$$

So

$$R_1 = [1 \ 0 \ 6 \ -8 \ 1].$$

Therefore, the reduced row echelon form of A is

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 6 & -8 & 1 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Now solve

$$\text{rref}(A)\mathbf{x} = \mathbf{0}.$$

This gives the system

$$\begin{cases} x_1 + 6x_3 - 8x_4 + x_5 = 0, \\ x_2 - 2x_3 + x_4 = 0. \end{cases}$$

The pivot variables are

$$x_1 \quad \text{and} \quad x_2.$$

The free variables are

$$x_3, \quad x_4, \quad x_5.$$

Let

$$x_3 = a, \quad x_4 = b, \quad x_5 = c,$$

where $a, b, c \in \mathbb{R}$.

From the first equation,

$$x_1 + 6x_3 - 8x_4 + x_5 = 0.$$

Solving for x_1 , we get

$$x_1 = -6x_3 + 8x_4 - x_5.$$

Substitute $x_3 = a$, $x_4 = b$, and $x_5 = c$:

$$x_1 = -6a + 8b - c.$$

From the second equation,

$$x_2 - 2x_3 + x_4 = 0.$$

Solving for x_2 , we get

$$x_2 = 2x_3 - x_4.$$

Substitute $x_3 = a$ and $x_4 = b$:

$$x_2 = 2a - b.$$

Thus every vector in the nullspace has the form

$$\mathbf{x} = \begin{bmatrix} -6a + 8b - c \\ 2a - b \\ a \\ b \\ c \end{bmatrix}.$$

Now separate this vector into a linear combination:

$$\mathbf{x} = a \begin{bmatrix} -6 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 8 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Therefore,

$$\text{Null}(A) = \text{span} \left\{ \begin{bmatrix} -6 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 8 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

So an explicit basis for the nullspace is

$$\{(-6, 2, 1, 0, 0), (8, -1, 0, 1, 0), (-1, 0, 0, 0, 1)\}.$$

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