Detailed Solutions to Worksheet #10

MATH031 – Spring 2026

Vector Spaces and Subspaces

Solution by Khoi Vo

Exercise 1

Decide in each case whether the given set is a subspace of the corresponding vector space.

Recall that a subset H of a vector space V is a subspace if:

1. the zero vector is in H,
2. H is closed under vector addition,
3. H is closed under scalar multiplication.

A useful fact is that if a set can be written as the span of some vectors, then it is automatically a subspace.

Exercise 1(a)

We are given

$$W = \{(s+3t,\ s-t,\ 2s-t,\ 4t) : s, t \text{ are scalars}\}.$$

We rewrite the vector by separating the terms involving s and the terms involving t:

$$(s+3t,\ s-t,\ 2s-t,\ 4t) = (s,\ s,\ 2s,\ 0) + (3t,\ -t,\ -t,\ 4t).$$

Factor out s and t:

$$(s+3t,\ s-t,\ 2s-t,\ 4t) = s(1,1,2,0) + t(3,-1,-1,4).$$

Therefore,

$$W = \text{Span}\{(1,1,2,0),(3,-1,-1,4)\}.$$

Since W is the span of vectors in $\mathbb{R}^4$, W is a subspace of $\mathbb{R}^4$.

$$\boxed{W \text{ is a subspace of } \mathbb{R}^4.}$$

Exercise 1(b)

We are given

$$H = \{(x, y, z, w) : x + y + z = 0 \text{ and } y + z + w = 0\}.$$

This set consists of all solutions to the homogeneous system

$$x + y + z = 0,$$

$$y + z + w = 0.$$

Because the right-hand side of each equation is 0, this is a homogeneous system. The solution set of a homogeneous linear system is always a subspace.

We can also write the system in matrix form:

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Thus H is the nullspace of the matrix

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}.$$

Since every nullspace is a subspace, H is a subspace of $\mathbb{R}^4$.

For an explicit description, solve the equations. From

$$x + y + z = 0,$$

we get

$$x = -y - z.$$

From

$$y + z + w = 0,$$

we get

$$w = -y - z.$$

Let

$$y = s, \qquad z = t.$$

Then

$$x = -s - t, \qquad w = -s - t.$$

So every vector in H has the form

$$(x, y, z, w) = (-s - t,\ s,\ t,\ -s - t).$$

Separate the parameters:

$$(-s - t,\ s,\ t,\ -s - t) = s(-1, 1, 0, -1) + t(-1, 0, 1, -1).$$

Therefore,

$$H = \text{Span}\{(-1, 1, 0, -1), (-1, 0, 1, -1)\}.$$

So H is a subspace.

$$\boxed{H \text{ is a subspace of } \mathbb{R}^4.}$$

Exercise 1(c)

We are given

$$P = \{(a, b, c) : a + b + c = 2\}.$$

To be a subspace of $\mathbb{R}^3$, the set must contain the zero vector

$$(0, 0, 0).$$

Check whether the zero vector satisfies the condition:

$$0 + 0 + 0 = 0.$$

But the condition for being in P is

$$a + b + c = 2.$$

Since

$$0 \neq 2,$$

we have

$$(0, 0, 0) \notin P.$$

Therefore, P is not a subspace.
We can also show that P is not closed under scalar multiplication. For example,

$$(2, 0, 0) \in P$$

because

$$2 + 0 + 0 = 2.$$

But multiplying this vector by 0 gives

$$0(2, 0, 0) = (0, 0, 0).$$

Since

$$(0, 0, 0) \notin P,$$

P is not closed under scalar multiplication.

$$\boxed{P \text{ is not a subspace of } \mathbb{R}^3.}$$

Exercise 2

The set of all continuous real-valued functions on a closed interval $[a, b]$ is denoted by

$$C[a, b].$$

It is considered as a subspace of the vector space of all functions

$$[a, b] \to \mathbb{R}.$$

Exercise 2(a)

We need to explain which facts from Calculus justify that $C[a, b]$ is a subspace.

The necessary facts are:

1. The zero function is continuous.
2. The sum of two continuous functions is continuous.
3. A scalar multiple of a continuous function is continuous.

First, the zero function is defined by

$$f(x) = 0$$

for every $x \in [a, b]$. This function is continuous on $[a, b]$. Hence the zero vector belongs to $C[a, b]$.

Second, suppose

$$f, g \in C[a, b].$$

This means f and g are continuous on $[a, b]$. From Calculus, the function

$$f + g$$

is also continuous on $[a, b]$. Therefore, $C[a, b]$ is closed under addition.

Third, suppose

$$f \in C[a, b]$$

and let $c \in \mathbb{R}$. Since f is continuous, the scalar multiple

$$cf$$

is also continuous on $[a, b]$. Therefore, $C[a, b]$ is closed under scalar multiplication.

Thus,

$$\boxed{C[a, b] \text{ is a subspace because continuity is preserved under sums and scalar multiples.}}$$

Exercise 2(b)

We need to show that

$$S = \{f \in C[a,b] : f(a) = f(b)\}$$

is a subspace of $C[a,b]$.

We check the three subspace properties.

Step 1: The zero function is in S

Let f be the zero function:

$$f(x) = 0$$

for all $x \in [a,b]$. Then

$$f(a) = 0$$

and

$$f(b) = 0.$$

Therefore,

$$f(a) = f(b).$$

So the zero function belongs to S.

Step 2: S is closed under addition

Suppose

$$f, g \in S.$$

Then

$$f(a) = f(b)$$

and

$$g(a) = g(b).$$

We need to show that $f + g \in S$.

Evaluate $f + g$ at the endpoints:

$$(f + g)(a) = f(a) + g(a),$$

and

$$(f + g)(b) = f(b) + g(b).$$

Because $f(a) = f(b)$ and $g(a) = g(b)$, we get

$$f(a) + g(a) = f(b) + g(b).$$

Therefore,

$$(f + g)(a) = (f + g)(b).$$

Also, since f and g are continuous, $f + g$ is continuous. Hence

$$f + g \in S.$$

Step 3: S is closed under scalar multiplication

Suppose

$$f \in S$$

and let $c \in \mathbb{R}$. Since $f \in S$, we know

$$f(a) = f(b).$$

Now consider cf. We have

$$(cf)(a) = cf(a)$$

and

$$(cf)(b) = cf(b).$$

Since $f(a) = f(b)$, multiplying both sides by c gives

$$cf(a) = cf(b).$$

Therefore,

$$(cf)(a) = (cf)(b).$$

Also, since f is continuous, cf is continuous. Hence

$$cf \in S.$$

Therefore, S is a subspace of $C[a, b]$.

$$\boxed{\{f \in C[a, b] : f(a) = f(b)\} \text{ is a subspace of } C[a, b].}$$

Exercise 3

Find an explicit description of Nul(A) by listing vectors that span it.

Recall that

$$\text{Nul}(A) = \{\mathbf{x} : A\mathbf{x} = \mathbf{0}\}.$$

So for each matrix, we solve the homogeneous system

$$A\mathbf{x} = \mathbf{0}.$$

Exercise 3(a)

We are given

$$A = \begin{bmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 4 & -2 \end{bmatrix}.$$

Let

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}.$$

We solve

$$A\mathbf{x} = \mathbf{0}.$$

This gives

$$\begin{bmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Therefore, the system is

$$x_1 + 3x_2 + 5x_3 = 0,$$

$$x_2 + 4x_3 - 2x_4 = 0.$$

From the second equation,

$$x_2 = -4x_3 + 2x_4.$$

Substitute this expression into the first equation:

$$x_1 + 3(-4x_3 + 2x_4) + 5x_3 = 0.$$

Simplify:

$$x_1 - 12x_3 + 6x_4 + 5x_3 = 0.$$

$$x_1 - 7x_3 + 6x_4 = 0.$$

Thus,

$$x_1 = 7x_3 - 6x_4.$$

The variables x_3 and x_4 are free. Let

$$x_3 = s, \qquad x_4 = t.$$

Then

$$x_2 = -4s + 2t$$

and

$$x_1 = 7s - 6t.$$

So

$$\mathbf{x} = \begin{bmatrix} 7s - 6t \\ -4s + 2t \\ s \\ t \end{bmatrix}.$$

Separate the parameters:

$$\mathbf{x} = s \begin{bmatrix} 7 \\ -4 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -6 \\ 2 \\ 0 \\ 1 \end{bmatrix}.$$

Therefore,

$$\text{Nul}(A) = \text{Span} \left\{ \begin{bmatrix} 7 \\ -4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -6 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Thus,

$$\boxed{\text{Nul}(A) = \text{Span}\{(7, -4, 1, 0), (-6, 2, 0, 1)\}.}$$

Exercise 3(b)

We are given

$$A = \begin{bmatrix} 1 & -2 & 0 & 4 & 0 \\ 0 & 0 & 1 & -9 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Let

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}.$$

We solve

$$A\mathbf{x} = \mathbf{0}.$$

Thus,

$$\begin{bmatrix} 1 & -2 & 0 & 4 & 0 \\ 0 & 0 & 1 & -9 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

This gives the system

$$x_1 - 2x_2 + 4x_4 = 0,$$

$$x_3 - 9x_4 = 0,$$

$$x_5 = 0.$$

From the first equation,

$$x_1 = 2x_2 - 4x_4.$$

From the second equation,

$$x_3 = 9x_4.$$

From the third equation,

$$x_5 = 0.$$

The free variables are x_2 and x_4. Let

$$x_2 = s, \qquad x_4 = t.$$

Then

$$x_1 = 2s - 4t,$$

$$x_3 = 9t,$$

and

$$x_5 = 0.$$

Therefore,

$$\mathbf{x} = \begin{bmatrix} 2s - 4t \\ s \\ 9t \\ t \\ 0 \end{bmatrix}.$$

Separate the parameters:

$$\mathbf{x} = s \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -4 \\ 0 \\ 9 \\ 1 \\ 0 \end{bmatrix}.$$

Therefore,

$$\text{Nul}(A) = \text{Span} \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 9 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

Thus,

$$\boxed{\text{Nul}(A) = \text{Span}\{(2, 1, 0, 0, 0), (-4, 0, 9, 1, 0)\}.}$$

Exercise 4

Determine whether each statement is true or false. Briefly justify each answer.

Exercise 4(i)

Statement:

If f is a function in the vector space V of all real-valued functions on $\mathbb{R}$

and if

$$f(t) = 0$$

for some t, then f is the zero vector in V.

This statement is false.

A function can equal zero at one input without being the zero function. The zero vector in the vector space of functions is the function that equals zero for every input.

For example, define

$$f(x) = x.$$

Then

$$f(0) = 0,$$

but

$$f(1) = 1.$$

So f is not the zero function.

$$\boxed{\text{False}}$$

Exercise 4(ii)

Statement:

A vector is any element of a vector space.

This statement is true.

In linear algebra, a vector means any element of a vector space. A vector does not have to be an arrow or an ordered list. Depending on the vector space, vectors can be ordered tuples, matrices, polynomials, functions, and other objects.

$$\boxed{\text{True}}$$

Exercise 4(iii)

Statement:

An arrow in three-dimensional space can be considered to be a vector.

This statement is true.

An arrow in three-dimensional space represents both direction and magnitude. This is one geometric interpretation of a vector in $\mathbb{R}^3$.

$$\boxed{\text{True}}$$

Exercise 4(iv)

Statement:

A subset H of a vector space is a subspace of V if the zero vector is in H.

This statement is false.

Containing the zero vector is necessary, but it is not enough. A subspace must also be closed under vector addition and scalar multiplication.

For example, let

$$H = \{0, 1\}$$

inside $\mathbb{R}$. This set contains the zero vector 0. However,

$$1 + 1 = 2,$$

and

$$2 \notin H.$$

Therefore, H is not closed under addition, so H is not a subspace.

$$\boxed{\text{False}}$$

Exercise 4(v)

Statement:

$\mathbb{R}^2$ is a subspace of $\mathbb{R}^3$.

This statement is false.

A subspace must first be a subset of the vector space. But elements of $\mathbb{R}^2$ are ordered pairs

$$(x, y),$$

while elements of $\mathbb{R}^3$ are ordered triples

$$(x, y, z).$$

Therefore, $\mathbb{R}^2$ is not literally a subset of $\mathbb{R}^3$.
However, a copy of $\mathbb{R}^2$, such as the xy-plane,

$$\{(x, y, 0) : x, y \in \mathbb{R}\},$$

is a subspace of $\mathbb{R}^3$.

$$\boxed{\text{False}}$$

Exercise 4(vi)

Statement:

The nullspace of A is the solution set of $A\mathbf{x} = \mathbf{0}$.

This statement is true.
By definition,

$$\text{Nul}(A) = \{\mathbf{x} : A\mathbf{x} = \mathbf{0}\}.$$

So the nullspace of A is exactly the solution set of the homogeneous equation

$$A\mathbf{x} = \mathbf{0}.$$

$$\boxed{\text{True}}$$

Exercise 4(vii)

Statement:

A nullspace is a vector space.

This statement is true.
The nullspace of a matrix is always a subspace of the domain. Since every subspace is itself a vector space, a nullspace is a vector space.

$$\boxed{\text{True}}$$

Exercise 4(viii)

Statement:

$$\text{The nullspace of an } m \times n \text{ matrix is in } \mathbb{R}^m.$$

This statement is false.
If A is an $m \times n$ matrix, then A multiplies vectors in $\mathbb{R}^n$. That is,

$$A\mathbf{x}$$

is defined when

$$\mathbf{x} \in \mathbb{R}^n.$$

The product $A\mathbf{x}$ lies in $\mathbb{R}^m$, but the nullspace consists of the input vectors $\mathbf{x}$ satisfying

$$A\mathbf{x} = \mathbf{0}.$$

Therefore,

$$\text{Nul}(A) \subseteq \mathbb{R}^n,$$

not $\mathbb{R}^m$.

$$\boxed{\text{False}}$$

Summary of Answers

Exercise 1

Part	Answer
(a)	Subspace
(b)	Subspace
(c)	Not a subspace

Exercise 2

$$\boxed{C[a,b] \text{ is a subspace because continuous functions are closed under addition and scalar multiplication.}}$$

$$\boxed{\{f \in C[a,b] : f(a) = f(b)\} \text{ is a subspace of } C[a,b].}$$

Exercise 3

$$\boxed{\text{Nul}\begin{bmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 4 & -2 \end{bmatrix} = \text{Span}\{(7,-4,1,0),(-6,2,0,1)\}}$$

$$\boxed{\text{Nul}\begin{bmatrix} 1 & -2 & 0 & 4 & 0 \\ 0 & 0 & 1 & -9 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \text{Span}\{(2,1,0,0,0),(-4,0,9,1,0)\}}$$

Exercise 4

Statement	Answer
(i)	False
(ii)	True
(iii)	True
(iv)	False
(v)	False
(vi)	True
(vii)	True
$(viii)$	False

Copyright and Academic Use Notice

This solution write-up is prepared for educational and study purposes.

The original worksheet questions, course title, formatting, and problem selection belong to the original instructor, department, or institution that created the worksheet. This document is an original explanatory solution based on those worksheet problems.

This solution may be used for personal study, tutoring, review, and learning support. It should not be submitted as a student's own independent work if the assignment is intended to be completed individually. Students should follow the academic integrity policies of their course and institution.

Suggested attribution:

> Based on MATH031 – Spring 2026 Worksheet #10, "Vector Spaces and Subspaces." Original worksheet by the course instructor or institution. Solution write-up prepared separately for educational use.