Detailed Solutions to Worksheet #11

MATH031 – Spring 2026

Column Spaces and Linear Transformations

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Exercise 1

We are given

$$A = \begin{bmatrix} -8 & -2 & -9 \\ 6 & 4 & 8 \\ 4 & 0 & 4 \end{bmatrix}$$

and

$$\mathbf{w} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}.$$

We need to determine whether $\mathbf{w}$ is in $\mathrm{Col}(A)$ and whether $\mathbf{w}$ is in $\mathrm{Nul}(A)$.

Part 1: Determine whether $\mathbf{w} \in \mathrm{Col}(A)$

Recall that $\mathbf{w}$ is in $\mathrm{Col}(A)$ if and only if the equation

$$A\mathbf{x} = \mathbf{w}$$

has a solution.
Let

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Then we solve

$$\begin{bmatrix} -8 & -2 & -9 \\ 6 & 4 & 8 \\ 4 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}.$$

This gives the system

$$-8x_1 - 2x_2 - 9x_3 = 2,$$

$$6x_1 + 4x_2 + 8x_3 = 1,$$

$$4x_1 + 4x_3 = -2.$$

We write the augmented matrix:

$$\left[\begin{array}{ccc|c} -8 & -2 & -9 & 2 \\ 6 & 4 & 8 & 1 \\ 4 & 0 & 4 & -2 \end{array}\right].$$

Row-reducing, we obtain

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & -\frac{1}{2} \\ 0 & 1 & \frac{1}{2} & \frac{13}{2} \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Since there is no contradiction, the system is consistent. Therefore, the equation

$$A\mathbf{x} = \mathbf{w}$$

has a solution.
Thus,

$$\boxed{\mathbf{w} \in \mathrm{Col}(A).}$$

In fact, from the row-reduced system,

$$x_1 + x_3 = -\frac{1}{2},$$

$$x_2 + \frac{1}{2}x_3 = \frac{13}{2}.$$

Let

$$x_3 = t.$$

Then

$$x_1 = -\frac{1}{2} - t,$$

and

$$x_2 = \frac{13}{2} - \frac{1}{2}t.$$

So one possible solution is obtained by choosing $t = 0$:

$$\mathbf{x} = \begin{bmatrix} -\frac{1}{2} \\ \frac{13}{2} \\ 0 \end{bmatrix}.$$

This confirms that $\mathbf{w}$ is a linear combination of the columns of A.

Part 2: Determine whether $\mathbf{w} \in \text{Nul}(A)$

Recall that

$$\mathbf{w} \in \text{Nul}(A)$$

if and only if

$$A\mathbf{w} = \mathbf{0}.$$

Compute

$$A\mathbf{w} = \begin{bmatrix} -8 & -2 & -9 \\ 6 & 4 & 8 \\ 4 & 0 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}.$$

Now calculate each entry.
First entry:

$$-8(2) - 2(1) - 9(-2) = -16 - 2 + 18 = 0.$$

Second entry:

$$6(2) + 4(1) + 8(-2) = 12 + 4 - 16 = 0.$$

Third entry:

$$4(2) + 0(1) + 4(-2) = 8 + 0 - 8 = 0.$$

Therefore,

$$A\mathbf{w} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Thus,

$$\boxed{\mathbf{w} \in \text{Nul}(A).}$$

Final Answer for Exercise 1

$$\boxed{\mathbf{w} \in \mathrm{Col}(A)}$$

and

$$\boxed{\mathbf{w} \in \mathrm{Nul}(A).}$$

Exercise 2

In each case, we need to find a matrix A such that the given subspace is equal to $\mathrm{Col}(A)$.

Recall that the column space of a matrix is the span of its columns. Therefore, if we can rewrite the given set as a span of vectors, then we can use those vectors as the columns of A.

Exercise 2(a)

We are given the set

$$\left\{ \begin{bmatrix} 2s+3t \\ r+s-2t \\ 4r+s \\ 3r-s-s \end{bmatrix} : r, s, t \in \mathbb{R} \right\}.$$

Notice that

$$3r - s - s = 3r - 2s.$$

So the vector can be written as

$$\begin{bmatrix} 2s+3t \\ r+s-2t \\ 4r+s \\ 3r-2s \end{bmatrix}.$$

Now separate the parameters r, s, and t.
The terms involving r are

$$\begin{bmatrix} 0 \\ r \\ 4r \\ 3r \end{bmatrix} = r \begin{bmatrix} 0 \\ 1 \\ 4 \\ 3 \end{bmatrix}.$$

The terms involving s are

$$\begin{bmatrix} 2s \\ s \\ s \\ -2s \end{bmatrix} = s \begin{bmatrix} 2 \\ 1 \\ 1 \\ -2 \end{bmatrix}.$$

The terms involving t are

$$\begin{bmatrix} 3t \\ -2t \\ 0 \\ 0 \end{bmatrix} = t \begin{bmatrix} 3 \\ -2 \\ 0 \\ 0 \end{bmatrix}.$$

Therefore,

$$\begin{bmatrix} 2s+3t \\ r+s-2t \\ 4r+s \\ 3r-2s \end{bmatrix} = r\begin{bmatrix} 0 \\ 1 \\ 4 \\ 3 \end{bmatrix} + s\begin{bmatrix} 2 \\ 1 \\ 1 \\ -2 \end{bmatrix} + t\begin{bmatrix} 3 \\ -2 \\ 0 \\ 0 \end{bmatrix}.$$

Thus the given subspace is

$$\text{Span}\left\{\begin{bmatrix} 0 \\ 1 \\ 4 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 0 \\ 0 \end{bmatrix}\right\}.$$

Therefore, one matrix A whose column space is this subspace is

$$A = \begin{bmatrix} 0 & 2 & 3 \\ 1 & 1 & -2 \\ 4 & 1 & 0 \\ 3 & -2 & 0 \end{bmatrix}.$$

Hence,

$$\boxed{A = \begin{bmatrix} 0 & 2 & 3 \\ 1 & 1 & -2 \\ 4 & 1 & 0 \\ 3 & -2 & 0 \end{bmatrix}.}$$

Exercise 2(b)

We are given the set

$$\left\{\begin{bmatrix} b-c \\ 2b+c+d \\ 5c-4d \\ d \end{bmatrix} : b, c, d \in \mathbb{R}\right\}.$$

We separate the parameters b, c, and d.
The terms involving b are

$$\begin{bmatrix} b \\ 2b \\ 0 \\ 0 \end{bmatrix} = b\begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}.$$

The terms involving c are

$$\begin{bmatrix} -c \\ c \\ 5c \\ 0 \end{bmatrix} = c\begin{bmatrix} -1 \\ 1 \\ 5 \\ 0 \end{bmatrix}.$$

The terms involving d are

$$\begin{bmatrix} 0 \\ d \\ -4d \\ d \end{bmatrix} = d \begin{bmatrix} 0 \\ 1 \\ -4 \\ 1 \end{bmatrix}.$$

Therefore,

$$\begin{bmatrix} b-c \\ 2b+c+d \\ 5c-4d \\ d \end{bmatrix} = b \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} -1 \\ 1 \\ 5 \\ 0 \end{bmatrix} + d \begin{bmatrix} 0 \\ 1 \\ -4 \\ 1 \end{bmatrix}.$$

Thus the given subspace is

$$\text{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 5 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -4 \\ 1 \end{bmatrix} \right\}.$$

Therefore, one matrix A whose column space is this subspace is

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & 1 \\ 0 & 5 & -4 \\ 0 & 0 & 1 \end{bmatrix}.$$

Hence,

$$\boxed{A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & 1 \\ 0 & 5 & -4 \\ 0 & 0 & 1 \end{bmatrix}.}$$

Exercise 3

Let $M_{2\times 2}$ be the vector space of all 2×2 matrices, and define

$$T : M_{2\times 2} \to M_{2\times 2}$$

by

$$T(A) = A + A^T.$$

Exercise 3(a)

We need to show that T is a linear transformation.

Recall that a transformation T is linear if for all matrices $A, B \in M_{2\times 2}$ and all scalars $c \in \mathbb{R}$,

$$T(A + B) = T(A) + T(B)$$

and

$$T(cA) = cT(A).$$

Let $A, B \in M_{2\times 2}$. Then

$$T(A + B) = (A + B) + (A + B)^T.$$

Using the transpose property

$$(A + B)^T = A^T + B^T,$$

we get

$$T(A + B) = A + B + A^T + B^T.$$

Rearrange the terms:

$$T(A + B) = (A + A^T) + (B + B^T).$$

But

$$T(A) = A + A^T$$

and

$$T(B) = B + B^T.$$

Therefore,

$$T(A + B) = T(A) + T(B).$$

Now let $c \in \mathbb{R}$. Then

$$T(cA) = cA + (cA)^T.$$

Using the transpose property

$$(cA)^T = cA^T,$$

we get

$$T(cA) = cA + cA^T.$$

Factor out c:

$$T(cA) = c(A + A^T).$$

Therefore,

$$T(cA) = cT(A).$$

Since both properties hold, T is linear.

$$\boxed{T \text{ is a linear transformation.}}$$

Exercise 3(b)

Let B be any element of $M_{2\times2}$ such that

$$B = B^T.$$

This means B is symmetric.
We need to find a matrix $A \in M_{2\times2}$ such that

$$T(A) = B.$$

Since

$$T(A) = A + A^T,$$

a simple choice is

$$A = \frac{1}{2}B.$$

Let us check this.
If

$$A = \frac{1}{2}B,$$

then

$$A^T = \left(\frac{1}{2}B\right)^T.$$

Using the transpose property,

$$A^T = \frac{1}{2}B^T.$$

Since $B = B^T$, we get

$$A^T = \frac{1}{2}B.$$

Therefore,

$$T(A) = A + A^T.$$

Substitute $A = \frac{1}{2}B$:

$$T(A) = \frac{1}{2}B + \frac{1}{2}B.$$

Thus,

$$T(A) = B.$$

So one matrix that works is

$$\boxed{A = \frac{1}{2}B.}$$

Exercise 3(c)

We need to show that the range of T is the set of all symmetric matrices.

That is, we want to show

$$\text{Range}(T) = \{B \in M_{2\times 2} : B = B^T\}.$$

We prove this using two inclusions.

First inclusion: Range(T) is contained in the set of symmetric matrices

Let

$$Y \in \text{Range}(T).$$

Then there exists some matrix $A \in M_{2\times 2}$ such that

$$Y = T(A).$$

By definition of T,

$$Y = A + A^T.$$

Now compute the transpose of Y:

$$Y^T = (A + A^T)^T.$$

Using the transpose property,

$$(A + A^T)^T = A^T + (A^T)^T.$$

Since

$$(A^T)^T = A,$$

we get

$$Y^T = A^T + A.$$

Because matrix addition is commutative,

$$A^T + A = A + A^T.$$

Therefore,

$$Y^T = Y.$$

Thus, every matrix in the range of T is symmetric.
So,

$$\text{Range}(T) \subseteq \{B \in M_{2\times 2} : B = B^T\}.$$

Second inclusion: Every symmetric matrix is in $\text{Range}(T)$

Now let

$$B \in M_{2\times 2}$$

be symmetric. That means

$$B = B^T.$$

From Exercise 3(b), choosing

$$A = \frac{1}{2}B$$

gives

$$T(A) = B.$$

Therefore, every symmetric matrix B is in the range of T.

So,

$$\{B \in M_{2\times 2} : B = B^T\} \subseteq \text{Range}(T).$$

Since both inclusions hold, we conclude that

$$\boxed{\text{Range}(T) = \{B \in M_{2\times 2} : B = B^T\}.}$$

In words, the range of T is exactly the set of all symmetric 2×2 matrices.

Exercise 3(d)

We need to describe

$$\ker(T).$$

Recall that the kernel of T is

$$\ker(T) = \{A \in M_{2\times 2} : T(A) = 0\}.$$

Since

$$T(A) = A + A^T,$$

we need

$$A + A^T = 0.$$

This means

$$A^T = -A.$$

So the kernel consists of all skew-symmetric 2×2 matrices.
Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Then

$$A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}.$$

Now compute

$$A + A^T = \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 2a & b+c \\ b+c & 2d \end{bmatrix}.$$

We want

$$A + A^T = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

So we need

$$2a = 0,$$

$$b + c = 0,$$

$$2d = 0.$$

Thus,

$$a = 0,$$

$$d = 0,$$

and

$$c = -b.$$

Therefore, matrices in the kernel have the form

$$A = \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}.$$

Let

$$b = t.$$

Then

$$A = \begin{bmatrix} 0 & t \\ -t & 0 \end{bmatrix} = t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Therefore,

$$\ker(T) = \left\{ \begin{bmatrix} 0 & t \\ -t & 0 \end{bmatrix} : t \in \mathbb{R} \right\}.$$

Equivalently,

$$\boxed{\ker(T) = \operatorname{Span}\left\{ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}.}$$

Summary of Answers

Exercise 1

$$\boxed{\mathbf{w} \in \mathrm{Col}(A)}$$

and

$$\boxed{\mathbf{w} \in \mathrm{Nul}(A).}$$

Exercise 2

For part (a), one possible matrix is

$$\boxed{A = \begin{bmatrix} 0 & 2 & 3 \\ 1 & 1 & -2 \\ 4 & 1 & 0 \\ 3 & -2 & 0 \end{bmatrix}.}$$

For part (b), one possible matrix is

$$\boxed{A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & 1 \\ 0 & 5 & -4 \\ 0 & 0 & 1 \end{bmatrix}.}$$

Exercise 3

$$\boxed{T(A) = A + A^T \text{ is linear.}}$$

If $B = B^T$, then

$$\boxed{A = \frac{1}{2}B}$$

satisfies

$$T(A) = B.$$

The range of T is

$$\boxed{\mathrm{Range}(T) = \{B \in M_{2\times 2} : B = B^T\}.}$$

The kernel of T is

$$\boxed{\ker(T) = \left\{ \begin{bmatrix} 0 & t \\ -t & 0 \end{bmatrix} : t \in \mathbb{R} \right\}.}$$

Equivalently,

$$\boxed{\ker(T) = \operatorname{Span}\left\{ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}.}$$

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