MATH 031 – Spring 2026 Worksheet #13

Detailed Solutions: Coordinate Vectors and Coordinate Systems

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Exercise 1

In each case, find the vector $\mathbf{x}$ determined by the given basis B and the corresponding B-coordinate vector.

Part (a)

$$B = \left\{ \begin{bmatrix} 3 \\ -5 \end{bmatrix}, \begin{bmatrix} -4 \\ 6 \end{bmatrix} \right\} \qquad \text{and} \qquad [\mathbf{x}]_B = \begin{bmatrix} 5 \\ 3 \end{bmatrix}.$$

Solution

The coordinate vector

$$[\mathbf{x}]_B = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

means that $\mathbf{x}$ is formed by taking 5 times the first basis vector and 3 times the second basis vector.

Therefore,

$$\mathbf{x} = 5 \begin{bmatrix} 3 \\ -5 \end{bmatrix} + 3 \begin{bmatrix} -4 \\ 6 \end{bmatrix}.$$

Now compute:

$$5 \begin{bmatrix} 3 \\ -5 \end{bmatrix} = \begin{bmatrix} 15 \\ -25 \end{bmatrix},$$

and

$$3 \begin{bmatrix} -4 \\ 6 \end{bmatrix} = \begin{bmatrix} -12 \\ 18 \end{bmatrix}.$$

Thus,

$$\mathbf{x} = \begin{bmatrix} 15 \\ -25 \end{bmatrix} + \begin{bmatrix} -12 \\ 18 \end{bmatrix} = \begin{bmatrix} 3 \\ -7 \end{bmatrix}.$$

$$\boxed{\mathbf{x} = \begin{bmatrix} 3 \\ -7 \end{bmatrix}}$$

Part (b)

$$B = \left\{ \begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 4 \\ -7 \\ 0 \end{bmatrix} \right\} \qquad \text{and} \qquad [\mathbf{x}]_B = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}.$$

Solution

The coordinate vector

$$[\mathbf{x}]_B = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}$$

means

$$\mathbf{x} = 3 \begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix} + 0 \begin{bmatrix} 5 \\ 2 \\ -2 \end{bmatrix} - 1 \begin{bmatrix} 4 \\ -7 \\ 0 \end{bmatrix}.$$

Now compute each term:

$$3 \begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ -12 \\ 9 \end{bmatrix},$$

$$0 \begin{bmatrix} 5 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

and

$$-1 \begin{bmatrix} 4 \\ -7 \\ 0 \end{bmatrix} = \begin{bmatrix} -4 \\ 7 \\ 0 \end{bmatrix}.$$

Therefore,

$$\mathbf{x} = \begin{bmatrix} 3 \\ -12 \\ 9 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -4 \\ 7 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -5 \\ 9 \end{bmatrix}.$$

$$\boxed{\mathbf{x} = \begin{bmatrix} -1 \\ -5 \\ 9 \end{bmatrix}}$$

Exercise 2

Now do the reverse: given B and $\mathbf{x}$, find the B-coordinates $[\mathbf{x}]_B$.

Part (a)

$$B = \left\{ \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ -5 \end{bmatrix} \right\} \qquad \text{and} \qquad \mathbf{x} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

Solution

We want to write $\mathbf{x}$ as a linear combination of the basis vectors:

$$\mathbf{x} = c_1 \begin{bmatrix} 1 \\ -3 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ -5 \end{bmatrix}.$$

That is,

$$\begin{bmatrix} -2 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -3 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ -5 \end{bmatrix}.$$

This gives the system

$$c_1 + 2c_2 = -2,$$

$$-3c_1 - 5c_2 = 1.$$

From the first equation,

$$c_1 = -2 - 2c_2.$$

Substitute this into the second equation:

$$-3(-2 - 2c_2) - 5c_2 = 1.$$

Simplify:

$$6 + 6c_2 - 5c_2 = 1.$$

Thus,

$$6 + c_2 = 1,$$

so

$$c_2 = -5.$$

Then

$$c_1 = -2 - 2(-5) = 8.$$

Therefore,

$$[\mathbf{x}]_B = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 8 \\ -5 \end{bmatrix}.$$

$$\boxed{[\mathbf{x}]_B = \begin{bmatrix} 8 \\ -5 \end{bmatrix}}$$

Part (b)

$$B = \left\{ \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \\ 9 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix} \right\} \qquad \text{and} \qquad \mathbf{x} = \begin{bmatrix} 8 \\ -9 \\ 6 \end{bmatrix}.$$

Solution

We want constants c_1, c_2, c_3 such that

$$\mathbf{x} = c_1 \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix} + c_2 \begin{bmatrix} -3 \\ 4 \\ 9 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix}.$$

So,

$$\begin{bmatrix} 8 \\ -9 \\ 6 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix} + c_2 \begin{bmatrix} -3 \\ 4 \\ 9 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix}.$$

This gives the system

$$c_1 - 3c_2 + 2c_3 = 8,$$

$$-c_1 + 4c_2 - 2c_3 = -9,$$

$$-3c_1 + 9c_2 + 4c_3 = 6.$$

We write this as an augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & -3 & 2 & 8 \\ -1 & 4 & -2 & -9 \\ -3 & 9 & 4 & 6 \end{array}\right].$$

Apply row operations.
First, use row 1 as the pivot row:

$$R_2 \leftarrow R_2 + R_1, \qquad R_3 \leftarrow R_3 + 3R_1.$$

Then

$$\left[\begin{array}{ccc|c} 1 & -3 & 2 & 8 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 10 & 30 \end{array}\right].$$

From the second row,

$$c_2 = -1.$$

From the third row,

$$10c_3 = 30,$$

so

$$c_3 = 3.$$

Use the first equation:

$$c_1 - 3c_2 + 2c_3 = 8.$$

Substitute $c_2 = -1$ and $c_3 = 3$:

$$c_1 - 3(-1) + 2(3) = 8.$$

Thus,

$$c_1 + 3 + 6 = 8.$$

So,

$$c_1 = -1.$$

Therefore,

$$[\mathbf{x}]_B = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix}.$$

$$\boxed{[\mathbf{x}]_B = \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix}}$$

Exercise 3

Let

$$p_1(t) = 1 + t^2,$$

$$p_2(t) = t - 3t^2,$$

and

$$p_3(t) = 1 + t - 3t^2.$$

Part (a)

Use coordinate vectors to show that these polynomials form a basis for P_2.

Solution

The space P_2 is the vector space of all polynomials of degree at most 2.

The standard basis for P_2 is

$$\{1, t, t^2\}.$$

We write each polynomial using coordinates relative to the standard basis.

First,

$$p_1(t) = 1 + t^2.$$

So,

$$[p_1] = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Second,

$$p_2(t) = t - 3t^2.$$

So,

$$[p_2] = \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}.$$

Third,

$$p_3(t) = 1 + t - 3t^2.$$

So,

$$[p_3] = \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}.$$

Now place these coordinate vectors as columns of a matrix:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & -3 & -3 \end{bmatrix}.$$

To show that the polynomials form a basis for P_2, we need to show that their coordinate vectors are linearly independent in $\mathbb{R}^3$.

Compute the determinant:

$$\det(A) = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & -3 & -3 \end{vmatrix}.$$

Expand along the first row:

$$\det(A) = 1 \begin{vmatrix} 1 & 1 \\ -3 & -3 \end{vmatrix} + 1 \begin{vmatrix} 0 & 1 \\ 1 & -3 \end{vmatrix}.$$

Now compute the two determinants:

$$\begin{vmatrix} 1 & 1 \\ -3 & -3 \end{vmatrix} = 1(-3) - 1(-3) = 0,$$

and

$$\begin{vmatrix} 0 & 1 \\ 1 & -3 \end{vmatrix} = 0(-3) - 1(1) = -1.$$

Therefore,

$$\det(A) = 0 + (-1) = -1.$$

Since

$$\det(A) \neq 0,$$

the coordinate vectors are linearly independent.

Since P_2 has dimension 3, and we have 3 linearly independent polynomials, the set

$$\{p_1, p_2, p_3\}$$

forms a basis for P_2.

$$\boxed{\{p_1, p_2, p_3\} \text{ is a basis for } P_2.}$$

Part (b)

If

$$B = \{p_1, p_2, p_3\},$$

find the polynomial q in P_2 such that

$$[q]_B = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}.$$

Solution

The coordinate vector

$$[q]_B = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

means

$$q(t) = -1p_1(t) + 1p_2(t) + 2p_3(t).$$

So,

$$q(t) = -p_1(t) + p_2(t) + 2p_3(t).$$

Now substitute the given polynomials:

$$q(t) = -(1 + t^2) + (t - 3t^2) + 2(1 + t - 3t^2).$$

Distribute:

$$q(t) = -1 - t^2 + t - 3t^2 + 2 + 2t - 6t^2.$$

Combine like terms.
Constants:

$$-1 + 2 = 1.$$

Linear terms:

$$t + 2t = 3t.$$

Quadratic terms:

$$-t^2 - 3t^2 - 6t^2 = -10t^2.$$

Therefore,

$$q(t) = 1 + 3t - 10t^2.$$

$$\boxed{q(t) = 1 + 3t - 10t^2}$$

Exercise 4

True or false? Briefly justify your answers.

Statement (i)

If $\mathbf{x}$ is in V and B contains n vectors, the B-coordinate vector of $\mathbf{x}$ is in $\mathbb{R}^n$.

Answer

$$\boxed{\text{True.}}$$

If

$$B = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\},$$

then every vector $\mathbf{x}$ in V can be written as

$$\mathbf{x} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_n\mathbf{b}_n.$$

The B-coordinate vector of $\mathbf{x}$ is

$$[\mathbf{x}]_B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}.$$

This is a vector in $\mathbb{R}^n$.
Therefore, the statement is true.

Statement (ii)

If B is the standard basis for $\mathbb{R}^n$, then the B-coordinate vector of $\mathbf{x}$ is $\mathbf{x}$ itself.

Answer

$$\boxed{\text{True.}}$$

The standard basis for $\mathbb{R}^n$ is

$$B = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}.$$

Every vector

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

can be written as

$$\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n.$$

Therefore,

$$[\mathbf{x}]_B = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \mathbf{x}.$$

So the statement is true.

Statement (iii)

If P_B is the change-of-coordinates matrix, then

$$[\mathbf{x}]_B = P_B\mathbf{x}.$$

Answer

$$\boxed{\text{False.}}$$

Usually, the change-of-coordinates matrix

$$P_B$$

has the basis vectors of B as its columns. That means

$$P_B[\mathbf{x}]_B = \mathbf{x}.$$

So P_B changes B-coordinates into standard coordinates.
To go from standard coordinates $\mathbf{x}$ to B-coordinates, we need

$$[\mathbf{x}]_B = P_B^{-1}\mathbf{x}.$$

Therefore, the statement

$$[\mathbf{x}]_B = P_B\mathbf{x}$$

is false in general.

Statement (iv)

The vectors P_3 and $\mathbb{R}^3$ are isomorphic.

Answer

$$\boxed{\text{False.}}$$

The vector space P_3 is the space of all polynomials of degree at most 3.
A general polynomial in P_3 has the form

$$a_0 + a_1 t + a_2 t^2 + a_3 t^3.$$

So a basis for P_3 is

$$\{1, t, t^2, t^3\}.$$

Therefore,

$$\dim(P_3) = 4.$$

However,

$$\dim(\mathbb{R}^3) = 3.$$

Two finite-dimensional vector spaces are isomorphic if and only if they have the same dimension.
Since

$$\dim(P_3) \neq \dim(\mathbb{R}^3),$$

we conclude that P_3 and $\mathbb{R}^3$ are not isomorphic.

Statement (v)

In general, vectors $\mathbf{u}_1, \ldots, \mathbf{u}_p$ are linearly independent in V if and only if their B-coordinates

$$[\mathbf{u}_1]_B, \ldots, [\mathbf{u}_p]_B$$

are linearly independent in $\mathbb{R}^n$.

Answer

$$\boxed{\text{True.}}$$

Coordinate mappings preserve linear independence.
Suppose

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_p\mathbf{u}_p = \mathbf{0}.$$

Taking B-coordinates of both sides gives

$$c_1[\mathbf{u}_1]_B + c_2[\mathbf{u}_2]_B + \cdots + c_p[\mathbf{u}_p]_B = [\mathbf{0}]_B.$$

Since

$$[\mathbf{0}]_B = \mathbf{0},$$

the same linear dependence relation appears among the coordinate vectors.

Therefore, the original vectors are linearly independent exactly when their coordinate vectors are linearly independent.

$$\boxed{\mathbf{u}_1, \ldots, \mathbf{u}_p \text{ are linearly independent in } V \iff [\mathbf{u}_1]_B, \ldots, [\mathbf{u}_p]_B \text{ are linearly independent in } \mathbb{R}^n.}$$

Final Answers Summary

1. (a)

$$\mathbf{x} = \begin{bmatrix} 3 \\ -7 \end{bmatrix}.$$

   (b)

$$\mathbf{x} = \begin{bmatrix} -1 \\ -5 \\ 9 \end{bmatrix}.$$

2. (a)

$$[\mathbf{x}]_B = \begin{bmatrix} 8 \\ -5 \end{bmatrix}.$$

   (b)

$$[\mathbf{x}]_B = \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix}.$$

3. (a) The polynomials

$$p_1(t) = 1 + t^2, \qquad p_2(t) = t - 3t^2, \qquad p_3(t) = 1 + t - 3t^2$$

   form a basis for P_2.

   (b)

$$q(t) = 1 + 3t - 10t^2.$$

4.

Statement	Answer
(i)	True
(ii)	True
(iii)	False
(iv)	False
(v)	True

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