MATH 031 Applied Linear Algebra
Worksheet #14 Detailed Solutions

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Exercise 1

Find a basis and the dimension of each subspace below.

Exercise 1(a)

We are given the subspace

$$S = \left\{ \begin{bmatrix} s-2t \\ s+t \\ 3t \end{bmatrix} : s, t \in \mathbb{R} \right\}.$$

We want to rewrite the vector in terms of the parameters s and t.

$$\begin{bmatrix} s-2t \\ s+t \\ 3t \end{bmatrix} = \begin{bmatrix} s \\ s \\ 0 \end{bmatrix} + \begin{bmatrix} -2t \\ t \\ 3t \end{bmatrix}.$$

Now factor out s and t:

$$\begin{bmatrix} s-2t \\ s+t \\ 3t \end{bmatrix} = s \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}.$$

Therefore,

$$S = \operatorname{span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} \right\}.$$

Now we check if these two vectors are linearly independent.
Suppose

$$c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

This gives

$$\begin{bmatrix} c_1 - 2c_2 \\ c_1 + c_2 \\ 3c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

From the third component,

$$3c_2 = 0.$$

So,

$$c_2 = 0.$$

Then from the first component,

$$c_1 - 2(0) = 0,$$

so

$$c_1 = 0.$$

Thus the only solution is

$$c_1 = 0, \qquad c_2 = 0.$$

Therefore, the two vectors are linearly independent.
So a basis for S is

$$\boxed{\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} \right\}}$$

and the dimension is

$$\boxed{\dim(S) = 2.}$$

Exercise 1(b)

We are given the subspace

$$S = \left\{ \begin{bmatrix} 2c \\ a - b \\ b - 3c \\ a + 2b \end{bmatrix} : a, b, c \in \mathbb{R} \right\}.$$

We rewrite the vector in terms of the parameters a, b, and c.

$$\begin{bmatrix} 2c \\ a - b \\ b - 3c \\ a + 2b \end{bmatrix} = \begin{bmatrix} 0 \\ a \\ 0 \\ a \end{bmatrix} + \begin{bmatrix} 0 \\ -b \\ b \\ 2b \end{bmatrix} + \begin{bmatrix} 2c \\ 0 \\ -3c \\ 0 \end{bmatrix}.$$

Now factor out a, b, and c:

$$\begin{bmatrix} 2c \\ a-b \\ b-3c \\ a+2b \end{bmatrix} = a\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} + b\begin{bmatrix} 0 \\ -1 \\ 1 \\ 2 \end{bmatrix} + c\begin{bmatrix} 2 \\ 0 \\ -3 \\ 0 \end{bmatrix}.$$

Therefore,

$$S = \operatorname{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -3 \\ 0 \end{bmatrix} \right\}.$$

Now we check whether these vectors are linearly independent.
Suppose

$$c_1\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 0 \\ -1 \\ 1 \\ 2 \end{bmatrix} + c_3\begin{bmatrix} 2 \\ 0 \\ -3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

This gives

$$\begin{bmatrix} 2c_3 \\ c_1-c_2 \\ c_2-3c_3 \\ c_1+2c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

From the first component,

$$2c_3 = 0.$$

Thus,

$$c_3 = 0.$$

From the third component,

$$c_2 - 3c_3 = 0.$$

Since $c_3 = 0$, this becomes

$$c_2 = 0.$$

From the second component,

$$c_1 - c_2 = 0.$$

Since $c_2 = 0$, this gives

$$c_1 = 0.$$

Therefore, the only solution is

$$c_1 = c_2 = c_3 = 0.$$

So the three vectors are linearly independent.
Thus a basis for S is

$$\boxed{\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -3 \\ 0 \end{bmatrix} \right\}}$$

and the dimension is

$$\boxed{\dim(S) = 3.}$$

Exercise 2

Find the dimensions of $\mathrm{Nul}(A)$, $\mathrm{Col}(A)$, and $\mathrm{Row}(A)$ in each case.

Matrix A

We are given

$$A = \begin{bmatrix} 1 & -6 & 9 & 0 & -2 \\ 0 & 1 & 2 & -4 & 5 \\ 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

This matrix is already in row echelon form.
The pivot columns are columns

$$1, \ 2, \ \text{and } 4.$$

Therefore, the number of pivot columns is

$$3.$$

So the rank is

$$\mathrm{rank}(A) = 3.$$

The dimension of the column space is the rank:

$$\dim(\mathrm{Col}(A)) = 3.$$

The dimension of the row space is also the rank:

$$\dim(\text{Row}(A)) = 3.$$

Since A has 5 columns, there are 5 variables in the equation

$$A\mathbf{x} = \mathbf{0}.$$

By the Rank Theorem,

$$\dim(\text{Nul}(A)) = \text{number of columns} - \text{rank}(A).$$

Thus,

$$\dim(\text{Nul}(A)) = 5 - 3 = 2.$$

Therefore,

$$\boxed{\dim(\text{Nul}(A)) = 2, \quad \dim(\text{Col}(A)) = 3, \quad \dim(\text{Row}(A)) = 3.}$$

Matrix B

We are given

$$B = \begin{bmatrix} 1 & 3 & -4 & 2 & -1 & 6 \\ 0 & 0 & 1 & -3 & 7 & 0 \\ 0 & 0 & 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

This matrix is already in row echelon form.
The pivot columns are columns

$$1,\ 3,\ \text{and } 4.$$

So the rank is

$$\text{rank}(B) = 3.$$

The dimension of the column space is the rank:

$$\dim(\text{Col}(B)) = 3.$$

The dimension of the row space is also the rank:

$$\dim(\text{Row}(B)) = 3.$$

Since B has 6 columns, there are 6 variables in the equation

$$B\mathbf{x} = \mathbf{0}.$$

By the Rank Theorem,

$$\dim(\text{Nul}(B)) = \text{number of columns} - \text{rank}(B).$$

Thus,

$$\dim(\text{Nul}(B)) = 6 - 3 = 3.$$

Therefore,

$$\boxed{\dim(\text{Nul}(B)) = 3, \quad \dim(\text{Col}(B)) = 3, \quad \dim(\text{Row}(B)) = 3.}$$

Matrix C

We are given

$$C = \begin{bmatrix} 1 & 0 & 9 & 5 \\ 0 & 0 & 1 & 4 \end{bmatrix}.$$

This matrix is already in row echelon form.
The pivot columns are columns

$$1 \quad \text{and} \quad 3.$$

Therefore,

$$\text{rank}(C) = 2.$$

The dimension of the column space is the rank:

$$\dim(\text{Col}(C)) = 2.$$

The dimension of the row space is also the rank:

$$\dim(\text{Row}(C)) = 2.$$

Since C has 4 columns, there are 4 variables in the equation

$$C\mathbf{x} = \mathbf{0}.$$

By the Rank Theorem,

$$\dim(\text{Nul}(C)) = \text{number of columns} - \text{rank}(C).$$

Thus,

$$\dim(\text{Nul}(C)) = 4 - 2 = 2.$$

Therefore,

$$\boxed{\dim(\text{Nul}(C)) = 2, \quad \dim(\text{Col}(C)) = 2, \quad \dim(\text{Row}(C)) = 2.}$$

Exercise 3

Compute the determinants of

$$A = \begin{bmatrix} 3 & 0 & 4 \\ 2 & 3 & 2 \\ 0 & 5 & -1 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 0 & 4 & 1 \\ 5 & -3 & 0 \\ 2 & 4 & 1 \end{bmatrix}.$$

Are A and B invertible?

Determinant of A

We compute

$$\det(A) = \begin{vmatrix} 3 & 0 & 4 \\ 2 & 3 & 2 \\ 0 & 5 & -1 \end{vmatrix}.$$

Using cofactor expansion along the first row,

$$\det(A) = 3 \begin{vmatrix} 3 & 2 \\ 5 & -1 \end{vmatrix} - 0 \begin{vmatrix} 2 & 2 \\ 0 & -1 \end{vmatrix} + 4 \begin{vmatrix} 2 & 3 \\ 0 & 5 \end{vmatrix}.$$

The middle term is zero because its coefficient is 0.
Now compute the 2×2 determinants:

$$\begin{vmatrix} 3 & 2 \\ 5 & -1 \end{vmatrix} = 3(-1) - 2(5) = -3 - 10 = -13.$$

Also,

$$\begin{vmatrix} 2 & 3 \\ 0 & 5 \end{vmatrix} = 2(5) - 3(0) = 10.$$

Therefore,

$$\det(A) = 3(-13) + 4(10).$$

So,

$$\det(A) = -39 + 40 = 1.$$

Thus,

$$\boxed{\det(A) = 1.}$$

Since

$$\det(A) \neq 0,$$

the matrix A is invertible.

$$\boxed{A \text{ is invertible.}}$$

Determinant of B

We compute

$$\det(B) = \begin{vmatrix} 0 & 4 & 1 \\ 5 & -3 & 0 \\ 2 & 4 & 1 \end{vmatrix}.$$

Using cofactor expansion along the first row,

$$\det(B) = 0 \begin{vmatrix} -3 & 0 \\ 4 & 1 \end{vmatrix} - 4 \begin{vmatrix} 5 & 0 \\ 2 & 1 \end{vmatrix} + 1 \begin{vmatrix} 5 & -3 \\ 2 & 4 \end{vmatrix}.$$

The first term is zero because its coefficient is 0.
Now compute the 2×2 determinants:

$$\begin{vmatrix} 5 & 0 \\ 2 & 1 \end{vmatrix} = 5(1) - 0(2) = 5.$$

Also,

$$\begin{vmatrix} 5 & -3 \\ 2 & 4 \end{vmatrix} = 5(4) - (-3)(2) = 20 + 6 = 26.$$

Therefore,

$$\det(B) = -4(5) + 1(26).$$

So,

$$\det(B) = -20 + 26 = 6.$$

Thus,

$$\boxed{\det(B) = 6.}$$

Since

$$\det(B) \neq 0,$$

the matrix B is invertible.

$$\boxed{B \text{ is invertible.}}$$

Exercise 4

True or false? Briefly justify your answers.

Exercise 4(i)

Statement:

> The number of pivot columns of a matrix equals the dimension of its column space.

This statement is true.

The pivot columns of a matrix form a basis for the column space. Since the dimension of a vector space is the number of vectors in a basis, the number of pivot columns equals the dimension of the column space.

$$\boxed{\text{True}}$$

Exercise 4(ii)

Statement:

> The number of variables in the equation $A\mathbf{x} = \mathbf{0}$ equals the nullity of A.

This statement is false.

The number of variables in $A\mathbf{x} = \mathbf{0}$ equals the number of columns of A.

However, the nullity of A is the dimension of $\text{Nul}(A)$, which equals the number of free variables, not the total number of variables.

For example, if

$$A = I_2,$$

then $A\mathbf{x} = \mathbf{0}$ has 2 variables, but the only solution is

$$\mathbf{x} = \mathbf{0}.$$

So,

$$\text{nullity}(A) = 0.$$

Thus, the number of variables is not necessarily equal to the nullity.

$$\boxed{\text{False}}$$

Exercise 4(iii)

Statement:

$$\text{The dimension of the vector space } \mathbb{P}_4 \text{ is } 4.$$

This statement is false.
The vector space $\mathbb{P}_4$ consists of all polynomials of degree at most 4.
A standard basis for $\mathbb{P}_4$ is

$$\{1, t, t^2, t^3, t^4\}.$$

There are 5 vectors in this basis.
Therefore,

$$\dim(\mathbb{P}_4) = 5.$$

So the statement is false.

$$\boxed{\text{False}}$$

Exercise 4(iv)

Statement:

$$\text{The dimensions of } \operatorname{Row}(A) \text{ and } \operatorname{Col}(A) \text{ are the same, even if } A \text{ is not square.}$$

This statement is true.
The dimension of the row space is called the row rank.
The dimension of the column space is called the column rank.
A fundamental theorem in linear algebra says that

$$\text{row rank} = \text{column rank}.$$

Both are equal to the rank of the matrix.
Therefore,

$$\dim(\operatorname{Row}(A)) = \dim(\operatorname{Col}(A)).$$

This is true even when A is not square.

$$\boxed{\text{True}}$$

Exercise 4(v)

Statement:

$$\text{If } \mathbf{v}_1, \ldots, \mathbf{v}_p \text{ span a finite-dimensional vector space } V,$$

and if T is a set of more than p vectors in V, then T is linearly dependent.

This statement is true.

Since

$$\mathbf{v}_1, \ldots, \mathbf{v}_p$$

span V, the dimension of V is at most p:

$$\dim(V) \leq p.$$

In a finite-dimensional vector space, any set containing more vectors than the dimension of the space must be linearly dependent.

Since T has more than p vectors, and

$$p \geq \dim(V),$$

the set T has more vectors than the dimension of V.

Therefore, T must be linearly dependent.

$$\boxed{\text{True}}$$

Exercise 4(vi)

Statement:

$$\text{A vector space is infinite-dimensional if it is spanned by an infinite set.}$$

This statement is false.

A finite-dimensional vector space can also be spanned by an infinite set.

For example, $\mathbb{R}^2$ is finite-dimensional because

$$\dim(\mathbb{R}^2) = 2.$$

However, $\mathbb{R}^2$ can be spanned by an infinite set, such as

$$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \ldots \right\}.$$

This infinite set still spans $\mathbb{R}^2$, but $\mathbb{R}^2$ is not infinite-dimensional.

A vector space is infinite-dimensional only if it cannot be spanned by any finite set.

Thus,

$$\boxed{\text{False}}$$

Final Answers

$$\boxed{\text{Exercise 1(a): Basis } = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} \right\}, \quad \dim = 2.}$$

$$\boxed{\text{Exercise 1(b): Basis } = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -3 \\ 0 \end{bmatrix} \right\}, \quad \dim = 3.}$$

$$\boxed{\begin{aligned} \text{For } A: \quad & \dim(\text{Nul}(A)) = 2, \\ & \dim(\text{Col}(A)) = 3, \\ & \dim(\text{Row}(A)) = 3. \end{aligned}}$$

$$\boxed{\begin{aligned} \text{For } B: \quad & \dim(\text{Nul}(B)) = 3, \\ & \dim(\text{Col}(B)) = 3, \\ & \dim(\text{Row}(B)) = 3. \end{aligned}}$$

$$\boxed{\begin{aligned} \text{For } C: \quad & \dim(\text{Nul}(C)) = 2, \\ & \dim(\text{Col}(C)) = 2, \\ & \dim(\text{Row}(C)) = 2. \end{aligned}}$$

$$\boxed{\det(A) = 1, \qquad A \text{ is invertible.}}$$

$$\boxed{\det(B) = 6, \qquad B \text{ is invertible.}}$$

Exercise 4: (i) True, (ii) False, (iii) False, (iv) True, (v) True, (vi) False.

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