MATH 031 – Spring 2026
Worksheet #8 Detailed Solutions

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Topics

Matrix algebra and invertible matrices.

Exercise 1

Which of the matrices below are invertible? Find their inverses.

$$S_\lambda = \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}, \qquad R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

For a 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

we know that

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$$

provided that

$$ad - bc \neq 0.$$

(a)

$$S_\lambda = \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix}.$$

Its determinant is

$$\det(S_\lambda) = 1(1) - 0(\lambda) = 1.$$

Since the determinant is nonzero, S_λ is invertible for every real number λ. Using the inverse formula,

$$S_\lambda^{-1} = \frac{1}{1} \begin{bmatrix} 1 & -\lambda \\ 0 & 1 \end{bmatrix}.$$

Therefore,

$$\boxed{S_\lambda^{-1} = \begin{bmatrix} 1 & -\lambda \\ 0 & 1 \end{bmatrix}}$$

Geometrically, S_λ is a shear matrix. It sends

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x + \lambda y \\ y \end{bmatrix}.$$

So the y-coordinate stays the same, while the x-coordinate is shifted by an amount depending on y. The inverse shear simply uses $-\lambda$ to undo the original shear.

(b)

$$B = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}.$$

Its determinant is

$$\det(B) = 1(4) - (-2)(-2) = 4 - 4 = 0.$$

Since the determinant is zero, B is not invertible.

$$\boxed{B \text{ is not invertible.}}$$

Geometrically, the second row is a multiple of the first row:

$$[-2 \quad 4] = -2[1 \quad -2].$$

So the matrix collapses the plane onto a line. Because information is lost, there is no inverse transformation.

(c)

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Its determinant is

$$\det(R_\theta) = \cos\theta\cos\theta - (-\sin\theta)(\sin\theta).$$

Thus,

$$\det(R_\theta) = \cos^2\theta + \sin^2\theta = 1.$$

Since the determinant is nonzero, R_θ is invertible for every θ.
Using the inverse formula,

$$R_\theta^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}.$$

Therefore,

$$\boxed{R_\theta^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}}$$

Geometrically, R_θ rotates vectors counterclockwise by angle θ. Its inverse rotates vectors clockwise by angle θ, or equivalently counterclockwise by angle $-\theta$.

So

$$\boxed{R_\theta^{-1} = R_{-\theta}.}$$

Exercise 2

Solve the three linear systems simultaneously:

$$\begin{bmatrix} 1 & 2 \\ 5 & 12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 2 \\ 5 & 12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \end{bmatrix},$$

and

$$\begin{bmatrix} 1 & 2 \\ 5 & 12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}.$$

Let

$$A = \begin{bmatrix} 1 & 2 \\ 5 & 12 \end{bmatrix}.$$

We want to solve three systems with the same coefficient matrix. Therefore, we can place all right-hand sides into one augmented matrix:

$$\left[\begin{array}{cc|ccc} 1 & 2 & -1 & 1 & 2 \\ 5 & 12 & 3 & -5 & 6 \end{array}\right].$$

Now row reduce.

First, replace Row 2 by Row 2 minus 5 Row 1:

$$R_2 \leftarrow R_2 - 5R_1.$$

This gives

$$\left[\begin{array}{cc|ccc} 1 & 2 & -1 & 1 & 2 \\ 0 & 2 & 8 & -10 & -4 \end{array}\right].$$

Now divide Row 2 by 2:

$$R_2 \leftarrow \frac{1}{2}R_2.$$

So

$$\left[\begin{array}{cc|ccc} 1 & 2 & -1 & 1 & 2 \\ 0 & 1 & 4 & -5 & -2 \end{array}\right].$$

Now eliminate the 2 above the pivot:

$$R_1 \leftarrow R_1 - 2R_2.$$

Then

$$\left[\begin{array}{cc|ccc} 1 & 0 & -9 & 11 & 6 \\ 0 & 1 & 4 & -5 & -2 \end{array}\right].$$

Therefore, the three solutions are:

$$\boxed{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -9 \\ 4 \end{bmatrix}}$$

for the first system,

$$\boxed{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 11 \\ -5 \end{bmatrix}}$$

for the second system, and

$$\boxed{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \end{bmatrix}}$$

for the third system.

Exercise 3

Find all possible choices of c for which

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 9 & c \\ 1 & c & 3 \end{bmatrix}$$

is singular. In the cases where A is invertible, find A^{-1}.
Start with the augmented matrix:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 9 & c & 0 & 1 & 0 \\ 1 & c & 3 & 0 & 0 & 1 \end{array}\right].$$

Subtract Row 1 from Rows 2 and 3:

$$R_2 \leftarrow R_2 - R_1, \qquad R_3 \leftarrow R_3 - R_1.$$

Then

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 8 & c-1 & -1 & 1 & 0 \\ 0 & c-1 & 2 & -1 & 0 & 1 \end{array}\right].$$

Now the lower-right 2×2 block of the coefficient side is

$$\begin{bmatrix} 8 & c-1 \\ c-1 & 2 \end{bmatrix}.$$

The matrix A is invertible exactly when this block is invertible. Its determinant is

$$8(2) - (c-1)^2 = 16 - (c-1)^2.$$

So A is singular when

$$16 - (c-1)^2 = 0.$$

Thus,

$$(c-1)^2 = 16.$$

Taking square roots,

$$c - 1 = \pm 4.$$

Therefore,

$$c = 5 \qquad \text{or} \qquad c = -3.$$

So

$$\boxed{A \text{ is singular when } c = 5 \text{ or } c = -3.}$$

Now assume

$$c \neq 5 \qquad \text{and} \qquad c \neq -3.$$

Let

$$D = 16 - (c-1)^2.$$

Simplify D:

$$D = 16 - (c^2 - 2c + 1) = 15 + 2c - c^2.$$

So

$$D = -c^2 + 2c + 15.$$

For $c \neq 5, -3$, we have $D \neq 0$.

We can compute the inverse using cofactors. For

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 9 & c \\ 1 & c & 3 \end{bmatrix},$$

the cofactor matrix is

$$C = \begin{bmatrix} 27 - c^2 & c - 3 & c - 9 \\ c - 3 & 2 & 1 - c \\ c - 9 & 1 - c & 8 \end{bmatrix}.$$

Since A is symmetric, the cofactor matrix is also symmetric, so the adjugate is the same matrix:

$$\operatorname{adj}(A) = \begin{bmatrix} 27 - c^2 & c - 3 & c - 9 \\ c - 3 & 2 & 1 - c \\ c - 9 & 1 - c & 8 \end{bmatrix}.$$

The determinant is

$$\det(A) = 16 - (c - 1)^2 = -c^2 + 2c + 15.$$

Therefore,

$$\boxed{A^{-1} = \frac{1}{-c^2 + 2c + 15} \begin{bmatrix} 27 - c^2 & c - 3 & c - 9 \\ c - 3 & 2 & 1 - c \\ c - 9 & 1 - c & 8 \end{bmatrix}}$$

for

$$\boxed{c \neq 5, -3.}$$

Exercise 4

True or false? Briefly justify each answer.

(i)

If $AB = AC$ and A is invertible, then $B = C$.

This statement is true.

Since A is invertible, multiply both sides on the left by A^{-1}:

$$A^{-1}AB = A^{-1}AC.$$

Since $A^{-1}A = I$, we get

$$IB = IC.$$

Therefore,

$$B = C.$$

So

$$\boxed{\text{True.}}$$

(ii)

If A and B are $n \times n$ and invertible, then $A^{-1}B^{-1}$ is the inverse of AB.

This statement is false.

The inverse of a product reverses the order:

$$(AB)^{-1} = B^{-1}A^{-1}.$$

To check:

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I.$$

But in general,

$$A^{-1}B^{-1}$$

is not the inverse of AB because matrix multiplication is not commutative.

So

$$\boxed{\text{False.}}$$

(iii)

If A is invertible, then the inverse of A^{-1} is A itself.

This statement is true.

Since

$$AA^{-1} = I$$

and

$$A^{-1}A = I,$$

the matrix that undoes A^{-1} is A.

Therefore,

$$(A^{-1})^{-1} = A.$$

So

$$\boxed{\text{True.}}$$

(iv)

If A is an invertible $n \times n$ matrix, the equation $Ax = b$ is consistent for each b in $\mathbb{R}^n$.

This statement is true.

If A is invertible, then for every $b \in \mathbb{R}^n$, the solution is

$$x = A^{-1}b.$$

Therefore, a solution always exists.

So

$$\boxed{\text{True.}}$$

(v)

If A can be row reduced to the identity matrix, then A must be invertible.

This statement is true.

If A row reduces to I, then every column has a pivot. Therefore, A is row equivalent to the identity matrix, so A is invertible.

Equivalently, the row reduction process gives

$$A \sim I,$$

which means A can be undone by elementary row operations.

Thus,

$$\boxed{\text{True.}}$$

Exercise 5

Let

$$A = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} 6 & 2 \\ 2 & 4 \end{bmatrix}, \qquad C = \begin{bmatrix} 4 & -2 \\ -6 & 3 \end{bmatrix}.$$

Solve, if possible, the matrix equations:

$$\text{(a) } AX + B = C$$

and

$$\text{(b) } XA + B = C.$$

First compute

$$C - B = \begin{bmatrix} 4 & -2 \\ -6 & 3 \end{bmatrix} - \begin{bmatrix} 6 & 2 \\ 2 & 4 \end{bmatrix}.$$

So

$$C - B = \begin{bmatrix} -2 & -4 \\ -8 & -1 \end{bmatrix}.$$

Let

$$D = C - B = \begin{bmatrix} -2 & -4 \\ -8 & -1 \end{bmatrix}.$$

(a)

We solve

$$AX + B = C.$$

Subtract B from both sides:

$$AX = C - B.$$

Thus,

$$AX = D.$$

If A is invertible, then

$$X = A^{-1}D.$$

Compute A^{-1}. Since

$$A = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix},$$

we have

$$\det(A) = 5(2) - 3(3) = 10 - 9 = 1.$$

Thus,

$$A^{-1} = \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}.$$

Now compute

$$X = A^{-1}D.$$

So

$$X = \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} -2 & -4 \\ -8 & -1 \end{bmatrix}.$$

Multiply:

$$X = \begin{bmatrix} 2(-2) + (-3)(-8) & 2(-4) + (-3)(-1) \\ (-3)(-2) + 5(-8) & (-3)(-4) + 5(-1) \end{bmatrix}.$$

Simplify:

$$X = \begin{bmatrix} -4 + 24 & -8 + 3 \\ 6 - 40 & 12 - 5 \end{bmatrix}.$$

Therefore,

$$X = \begin{bmatrix} 20 & -5 \\ -34 & 7 \end{bmatrix}.$$

So

$$\boxed{X = \begin{bmatrix} 20 & -5 \\ -34 & 7 \end{bmatrix}}$$

(b)

Now solve

$$XA + B = C.$$

Subtract B from both sides:

$$XA = C - B.$$

Thus,

$$XA = D.$$

Since A is invertible, multiply both sides on the right by A^{-1}:

$$XAA^{-1} = DA^{-1}.$$

Thus,

$$X = DA^{-1}.$$

So

$$X = \begin{bmatrix} -2 & -4 \\ -8 & -1 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}.$$

Multiply:

$$X = \begin{bmatrix} (-2)(2) + (-4)(-3) & (-2)(-3) + (-4)(5) \\ (-8)(2) + (-1)(-3) & (-8)(-3) + (-1)(5) \end{bmatrix}.$$

Simplify:

$$X = \begin{bmatrix} -4+12 & 6-20 \\ -16+3 & 24-5 \end{bmatrix}.$$

Therefore,

$$X = \begin{bmatrix} 8 & -14 \\ -13 & 19 \end{bmatrix}.$$

So

$$\boxed{X = \begin{bmatrix} 8 & -14 \\ -13 & 19 \end{bmatrix}}$$

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