Detailed Solutions to Worksheet #9

Khoi Vo

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Worksheet #9: Invertible Matrices and Vector Spaces

Exercise 1

Determine whether the matrices below are invertible or not. Use as few calculations as possible.

Exercise 1(a)

$$A = \begin{pmatrix} 5 & 7 \\ -3 & -6 \end{pmatrix}$$

For a 2×2 matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

the matrix is invertible if and only if its determinant is nonzero.

So we compute:

$$\det(A) = (5)(-6) - (7)(-3)$$

$$\det(A) = -30 + 21 = -9$$

Since

$$\det(A) = -9 \neq 0,$$

the matrix is invertible.

$$\boxed{\text{The matrix is invertible.}}$$

Exercise 1(b)

$$A = \begin{pmatrix} 5 & 0 & 0 \\ -3 & -7 & 0 \\ 8 & 5 & -1 \end{pmatrix}$$

This matrix is a lower triangular matrix because all entries above the main diagonal are zero.

For a triangular matrix, the determinant is the product of the diagonal entries.

The diagonal entries are:

$$5, \quad -7, \quad -1$$

Thus,

$$\det(A) = 5(-7)(-1)$$

$$\det(A) = 35$$

Since

$$\det(A) = 35 \neq 0,$$

the matrix is invertible.

$$\boxed{\text{The matrix is invertible.}}$$

Exercise 1(c)

$$A = \begin{pmatrix} -7 & 0 & 4 \\ 3 & 0 & -1 \\ 2 & 0 & 9 \end{pmatrix}$$

Notice that the second column is completely zero:

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

If a square matrix has a zero column, then its columns are linearly dependent. A matrix with linearly dependent columns cannot be invertible.

Equivalently, a zero column means the determinant must be zero.

Therefore,

$$\det(A) = 0$$

and the matrix is not invertible.

$$\boxed{\text{The matrix is not invertible.}}$$

Exercise 2

Show that the linear transformations below are invertible and find their inverses.

Exercise 2(a)

$$T : \mathbb{R}^2 \to \mathbb{R}^2$$

is given by

$$T(x_1, x_2) = (-5x_1 + 9x_2,\ 4x_1 - 7x_2)$$

First, write the transformation as a matrix multiplication.

$$T(x) = Ax$$

where

$$A = \begin{pmatrix} -5 & 9 \\ 4 & -7 \end{pmatrix}$$

To show that T is invertible, we check whether A is invertible.

$$\det(A) = (-5)(-7) - (9)(4)$$

$$\det(A) = 35 - 36 = -1$$

Since

$$\det(A) = -1 \neq 0,$$

the matrix A is invertible. Therefore, the linear transformation T is invertible.
Now we find the inverse matrix.
For a 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

we have

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Here,

$$a = -5, \quad b = 9, \quad c = 4, \quad d = -7$$

So

$$A^{-1} = \frac{1}{-1} \begin{pmatrix} -7 & -9 \\ -4 & -5 \end{pmatrix}$$

Multiplying by -1, we get

$$A^{-1} = \begin{pmatrix} 7 & 9 \\ 4 & 5 \end{pmatrix}$$

Therefore,

$$T^{-1}(y_1, y_2) = \begin{pmatrix} 7 & 9 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

Thus,

$$T^{-1}(y_1, y_2) = (7y_1 + 9y_2,\ 4y_1 + 5y_2)$$

$$\boxed{T^{-1}(y_1, y_2) = (7y_1 + 9y_2,\ 4y_1 + 5y_2)}$$

Check

We can check by applying T to $T^{-1}(y_1, y_2)$.

Let

$$x_1 = 7y_1 + 9y_2$$

and

$$x_2 = 4y_1 + 5y_2$$

Then

$$T(x_1, x_2) = (-5x_1 + 9x_2,\ 4x_1 - 7x_2)$$

First component:

$$-5(7y_1 + 9y_2) + 9(4y_1 + 5y_2)$$

$$= -35y_1 - 45y_2 + 36y_1 + 45y_2$$

$$= y_1$$

Second component:

$$4(7y_1 + 9y_2) - 7(4y_1 + 5y_2)$$

$$= 28y_1 + 36y_2 - 28y_1 - 35y_2$$

$$= y_2$$

Therefore,

$$T(T^{-1}(y_1, y_2)) = (y_1, y_2)$$

so our inverse is correct.

Exercise 2(b)

$$T : \mathbb{R}^2 \to \mathbb{R}^2$$

is given by

$$T(x_1, x_2) = (6x_1 - 8x_2,\ -5x_1 + 7x_2)$$

The standard matrix of T is

$$A = \begin{pmatrix} 6 & -8 \\ -5 & 7 \end{pmatrix}$$

Now compute the determinant:

$$\det(A) = (6)(7) - (-8)(-5)$$

$$\det(A) = 42 - 40 = 2$$

Since

$$\det(A) = 2 \neq 0,$$

the matrix A is invertible. Therefore, T is invertible.
Now we find A^{-1}.
Using the formula

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix},$$

we get

$$A^{-1} = \frac{1}{2}\begin{pmatrix} 7 & 8 \\ 5 & 6 \end{pmatrix}$$

Therefore,

$$A^{-1} = \begin{pmatrix} \frac{7}{2} & 4 \\ \frac{5}{2} & 3 \end{pmatrix}$$

Thus,

$$T^{-1}(y_1, y_2) = \frac{1}{2}\begin{pmatrix} 7 & 8 \\ 5 & 6 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

So

$$T^{-1}(y_1, y_2) = \left(\frac{7y_1 + 8y_2}{2}, \frac{5y_1 + 6y_2}{2}\right)$$

$$\boxed{T^{-1}(y_1, y_2) = \left(\frac{7y_1 + 8y_2}{2}, \frac{5y_1 + 6y_2}{2}\right)}$$

Check

Let

$$x_1 = \frac{7y_1 + 8y_2}{2}$$

and

$$x_2 = \frac{5y_1 + 6y_2}{2}$$

Then

$$T(x_1, x_2) = (6x_1 - 8x_2,\ -5x_1 + 7x_2)$$

First component:

$$6\left(\frac{7y_1 + 8y_2}{2}\right) - 8\left(\frac{5y_1 + 6y_2}{2}\right)$$

$$= 3(7y_1 + 8y_2) - 4(5y_1 + 6y_2)$$

$$= 21y_1 + 24y_2 - 20y_1 - 24y_2$$

$$= y_1$$

Second component:

$$-5\left(\frac{7y_1+8y_2}{2}\right)+7\left(\frac{5y_1+6y_2}{2}\right)$$

$$=\frac{-35y_1-40y_2+35y_1+42y_2}{2}$$

$$=\frac{2y_2}{2}=y_2$$

Thus,

$$T(T^{-1}(y_1,y_2))=(y_1,y_2)$$

so the inverse is correct.

Exercise 3

True or false? Briefly justify your answers.

Statement (i)

If the equation $Ax = 0$ has only the trivial solution, then A is row equivalent to I_n.

$$\boxed{\text{True}}$$

If $Ax = 0$ has only the trivial solution, then the columns of A are linearly independent.

For an $n \times n$ matrix, having linearly independent columns means there is a pivot in every column. Since there are n columns, there are n pivots.

Thus, the reduced row echelon form of A must be the identity matrix I_n.

Therefore, A is row equivalent to I_n.

Statement (ii)

If the columns of A span $\mathbb{R}^n$, then the columns are linearly independent.

$$\boxed{\text{True, if } A \text{ is an } n \times n \text{ matrix.}}$$

Since this worksheet is about square $n \times n$ matrices, we interpret A as an $n \times n$ matrix.

If the columns of A span $\mathbb{R}^n$, then there is a pivot in every row.

For an $n \times n$ matrix, a pivot in every row also means a pivot in every column. Therefore, the columns are linearly independent.

$$\boxed{\text{True}}$$

Statement (iii)

If A is an $n \times n$ matrix, then $Ax = b$ has at least one solution for each $b \in \mathbb{R}^n$.

$$\boxed{\text{False}}$$

Not every $n \times n$ matrix is invertible.

The equation $Ax = b$ has a solution for every $b \in \mathbb{R}^n$ only when the columns of A span $\mathbb{R}^n$.

For example, consider

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Then

$$Ax = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}$$

This can never equal

$$b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

because the second component of Ax is always 0.

Thus, $Ax = b$ does not always have a solution for every b.

$$\boxed{\text{False}}$$

Statement (iv)

If $Ax = 0$ has a nontrivial solution, then A has fewer than n pivot positions.

$$\boxed{\text{True}}$$

A nontrivial solution to $Ax = 0$ means there is a solution other than the zero vector.

This happens exactly when there is at least one free variable.

If there is a free variable, then not every column has a pivot.

For an $n \times n$ matrix, this means A has fewer than n pivot positions.

$$\boxed{\text{True}}$$

Statement (v)

A 5×5 matrix may be invertible when its columns do not span $\mathbb{R}^5$.

$$\boxed{\text{False}}$$

By the Invertible Matrix Theorem, a 5×5 matrix is invertible if and only if its columns span $\mathbb{R}^5$.

Therefore, if the columns do not span $\mathbb{R}^5$, the matrix cannot be invertible.

$$\boxed{\text{False}}$$

Exercise 4

Let

$$V = \{(x, y) : x \geq 0 \text{ and } y \geq 0\}$$

This is the first quadrant in the xy-plane.

Exercise 4(a)

If u and v are in V, is $u + v$ in V? Why?

Let

$$u = (u_1, u_2)$$

and

$$v = (v_1, v_2)$$

Since $u \in V$, we know

$$u_1 \geq 0 \quad \text{and} \quad u_2 \geq 0$$

Since $v \in V$, we know

$$v_1 \geq 0 \quad \text{and} \quad v_2 \geq 0$$

Now add the vectors:

$$u + v = (u_1 + v_1,\ u_2 + v_2)$$

Because the sum of two nonnegative numbers is nonnegative, we have

$$u_1 + v_1 \geq 0$$

and

$$u_2 + v_2 \geq 0$$

Therefore,

$$u + v \in V$$

$$\boxed{\text{Yes, } u + v \text{ is in } V.}$$

Exercise 4(b)

Find a specific vector $u \in V$ and a specific scalar α such that $\alpha u \notin V$.

Choose

$$u = (1, 1)$$

Clearly,

$$1 \geq 0 \quad \text{and} \quad 1 \geq 0$$

so

$$u = (1, 1) \in V$$

Now choose the scalar

$$\alpha = -1$$

Then

$$\alpha u = -1(1, 1)$$

$$\alpha u = (-1, -1)$$

But

$$-1 < 0$$

so $(-1, -1)$ is not in the first quadrant.

Therefore,

$$\alpha u \notin V$$

This shows that V is not closed under scalar multiplication. Since a vector space must be closed under scalar multiplication, V is not a vector space.

$$\boxed{u = (1, 1), \quad \alpha = -1}$$

$$\boxed{V \text{ is not a vector space.}}$$

Exercise 5

Let

$$W = \{(x, y) : xy \geq 0\}$$

This is the union of the first and third quadrants in the xy-plane.

Exercise 5(a)

If $u \in W$ and α is any scalar, is $\alpha u \in W$? Why?

Let

$$u = (x, y)$$

Since $u \in W$, we know

$$xy \geq 0$$

Now multiply u by a scalar α:

$$\alpha u = (\alpha x, \alpha y)$$

To check whether $\alpha u \in W$, multiply the two coordinates:

$$(\alpha x)(\alpha y)$$

$$= \alpha^2 xy$$

Since

$$\alpha^2 \geq 0$$

for every real number α, and since

$$xy \geq 0,$$

we get

$$\alpha^2 xy \geq 0$$

Therefore,

$$\alpha u \in W$$

$$\boxed{\text{Yes, } W \text{ is closed under scalar multiplication.}}$$

Exercise 5(b)

Find specific vectors u and v in W such that $u + v \notin W$.

We need two vectors whose coordinate products are nonnegative, but whose sum has coordinate product negative.

Choose

$$u = (1, 2)$$

Then

$$xy = (1)(2) = 2 \geq 0$$

so

$$u \in W$$

Choose

$$v = (-2, -1)$$

Then

$$xy = (-2)(-1) = 2 \geq 0$$

so

$$v \in W$$

Now compute the sum:

$$u + v = (1, 2) + (-2, -1)$$

$$u + v = (-1, 1)$$

Now multiply the coordinates:

$$(-1)(1) = -1$$

Since

$$-1 < 0,$$

we have

$$u + v \notin W$$

Thus, W is not closed under vector addition. Since a vector space must be closed under vector addition, W is not a vector space.

$$\boxed{u = (1, 2), \quad v = (-2, -1)}$$

$$\boxed{u + v = (-1, 1) \notin W}$$

$$\boxed{W \text{ is not a vector space.}}$$

Final Summary

- In Exercise 1, we used determinants and structural observations to decide invertibility.
- In Exercise 2, we represented each linear transformation by a matrix and used inverse matrices to find the inverse transformations.
- In Exercise 3, we applied the Invertible Matrix Theorem and pivot concepts.
- In Exercise 4, the first quadrant is closed under addition but not scalar multiplication.
- In Exercise 5, the union of the first and third quadrants is closed under scalar multiplication but not vector addition.