MATH 031 – Spring 2026 Worksheet #12

Detailed Solutions: Linear Independence and Bases

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Exercise 1

Find a basis for the subspace

$$H = \{A \in M_{2\times 2} : A = A^T\}$$

of all symmetric 2×2 matrices.

Solution

Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Then

$$A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}.$$

The condition $A = A^T$ means

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}.$$

Therefore,

$$b = c.$$

So every symmetric 2×2 matrix has the form

$$A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}.$$

We can rewrite this matrix as a linear combination:

$$\begin{bmatrix} a & b \\ b & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Thus, a basis for H is

$$\boxed{\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.}$$

Explanation

These three matrices span all symmetric 2×2 matrices because any symmetric matrix can be written as a linear combination of them. They are also linearly independent because none of these matrices can be written as a linear combination of the others. Therefore, they form a basis.

So,

$$\dim(H) = 3.$$

Exercise 2

Determine in each case whether the given vectors are a basis for $\mathbb{R}^3$.

Part (a)

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Solution

There are three vectors in $\mathbb{R}^3$. To be a basis for $\mathbb{R}^3$, they must be linearly independent.

Place the vectors as columns:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

This matrix is upper triangular, and its determinant is

$$\det(A) = 1 \cdot 1 \cdot 1 = 1.$$

Since

$$\det(A) \neq 0,$$

the vectors are linearly independent.

Therefore, the vectors form a basis for $\mathbb{R}^3$.

$$\boxed{\text{Yes, this set is a basis for } \mathbb{R}^3.}$$

Part (b)

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Solution

This set contains the zero vector:

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Any set containing the zero vector is automatically linearly dependent. A basis must be linearly independent.

Therefore, this set cannot be a basis for $\mathbb{R}^3$.

$$\boxed{\text{No, this set is not a basis for } \mathbb{R}^3.}$$

Part (c)

$$\left\{ \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 9 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -3 \\ 5 \end{bmatrix} \right\}$$

Solution

This set contains four vectors in $\mathbb{R}^3$.

A basis for $\mathbb{R}^3$ must contain exactly three linearly independent vectors. Since this set has four vectors, it must be linearly dependent.

Therefore, this set cannot be a basis for $\mathbb{R}^3$.

$$\boxed{\text{No, this set is not a basis for } \mathbb{R}^3.}$$

Part (d)

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} -4 \\ -5 \\ 6 \end{bmatrix} \right\}$$

Solution

This set contains only two vectors in $\mathbb{R}^3$.

A basis for $\mathbb{R}^3$ must span all of $\mathbb{R}^3$. Two vectors cannot span all of $\mathbb{R}^3$.

Therefore, this set cannot be a basis for $\mathbb{R}^3$.

$$\boxed{\text{No, this set is not a basis for } \mathbb{R}^3.}$$

Exercise 3

Find bases for Nul(A), Col(A), and Row(A), where

$$A = \begin{bmatrix} -2 & 4 & -2 & -4 \\ 2 & -6 & -3 & 1 \\ -3 & 8 & 2 & -3 \end{bmatrix}.$$

Solution

We row-reduce A.

$$A = \begin{bmatrix} -2 & 4 & -2 & -4 \\ 2 & -6 & -3 & 1 \\ -3 & 8 & 2 & -3 \end{bmatrix}.$$

The reduced row echelon form is

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 6 & 5 \\ 0 & 1 & \frac{5}{2} & \frac{3}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The pivot columns are columns 1 and 2.

Basis for Nul(A)

To find Nul(A), solve

$$A\mathbf{x} = \mathbf{0}.$$

Using the reduced row echelon form, we have

$$\begin{bmatrix} 1 & 0 & 6 & 5 \\ 0 & 1 & \frac{5}{2} & \frac{3}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

This gives the equations

$$x_1 + 6x_3 + 5x_4 = 0,$$

and

$$x_2 + \frac{5}{2}x_3 + \frac{3}{2}x_4 = 0.$$

Thus,

$$x_1 = -6x_3 - 5x_4,$$

and

$$x_2 = -\frac{5}{2}x_3 - \frac{3}{2}x_4.$$

Let

$$x_3 = s, \qquad x_4 = t.$$

Then

$$x_1 = -6s - 5t,$$

and

$$x_2 = -\frac{5}{2}s - \frac{3}{2}t.$$

Therefore,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -6s - 5t \\ -\frac{5}{2}s - \frac{3}{2}t \\ s \\ t \end{bmatrix}.$$

Separate the parameters:

$$\mathbf{x} = s\begin{bmatrix} -6 \\ -\frac{5}{2} \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -5 \\ -\frac{3}{2} \\ 0 \\ 1 \end{bmatrix}.$$

So a basis for Nul(A) is

$$\boxed{\left\{ \begin{bmatrix} -6 \\ -\frac{5}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ -\frac{3}{2} \\ 0 \\ 1 \end{bmatrix} \right\}.}$$

To avoid fractions, we may also multiply each vector by 2. An equivalent basis is

$$\boxed{\left\{ \begin{bmatrix} -12 \\ -5 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -10 \\ -3 \\ 0 \\ 2 \end{bmatrix} \right\}.}$$

Basis for Col(A)

The pivot columns of the reduced row echelon form are columns 1 and 2. To find a basis for the column space of A, we use the corresponding columns of the original matrix A.

Column 1 of A is

$$\begin{bmatrix} -2 \\ 2 \\ -3 \end{bmatrix}.$$

Column 2 of A is

$$\begin{bmatrix} 4 \\ -6 \\ 8 \end{bmatrix}.$$

Therefore, a basis for Col(A) is

$$\boxed{\left\{ \begin{bmatrix} -2 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 4 \\ -6 \\ 8 \end{bmatrix} \right\}.}$$

Basis for Row(A)

A basis for the row space of A is given by the nonzero rows of the reduced row echelon form.

The nonzero rows are

$$\begin{bmatrix} 1 & 0 & 6 & 5 \end{bmatrix},$$

and

$$\begin{bmatrix} 0 & 1 & \frac{5}{2} & \frac{3}{2} \end{bmatrix}.$$

Therefore, a basis for Row(A) is

$$\boxed{\left\{ \begin{bmatrix} 1 & 0 & 6 & 5 \end{bmatrix}, \begin{bmatrix} 0 & 1 & \frac{5}{2} & \frac{3}{2} \end{bmatrix} \right\}.}$$

Summary

$$\boxed{\text{rank}(A) = 2.}$$

So,

$$\dim(\text{Col}(A)) = 2,$$

$$\dim(\text{Row}(A)) = 2,$$

and

$$\dim(\text{Nul}(A)) = 4 - 2 = 2.$$

Exercise 4

Find a basis for the set of vectors in $\mathbb{R}^3$ in the plane

$$x + 2y + z = 0.$$

Solution

We want all vectors

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3$$

satisfying

$$x + 2y + z = 0.$$

Solve for x:

$$x = -2y - z.$$

Let

$$y = s, \qquad z = t.$$

Then

$$x = -2s - t.$$

So every vector in the plane has the form

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2s - t \\ s \\ t \end{bmatrix}.$$

Separate the parameters:

$$\begin{bmatrix} -2s - t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

Therefore, a basis for the plane is

$$\boxed{\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}.}$$

Explanation

The plane $x + 2y + z = 0$ passes through the origin, so it is a subspace of $\mathbb{R}^3$. Since it is a plane through the origin, it should have dimension 2. Our basis contains two linearly independent vectors, so this is correct.

Exercise 5

True or false? Briefly justify your answers.

(i)

A single vector by itself is always linearly dependent.

Answer

$$\boxed{\text{False.}}$$

A set containing one vector $\{\mathbf{v}\}$ is linearly dependent only if

$$\mathbf{v} = \mathbf{0}.$$

If

$$\mathbf{v} \neq \mathbf{0},$$

then $\{\mathbf{v}\}$ is linearly independent.

(ii)

A linearly independent set in a subspace H is a basis for H.

Answer

$$\boxed{\text{False.}}$$

A basis for H must be both linearly independent and spanning.
A linearly independent set in H may fail to span all of H. For example, in $\mathbb{R}^3$,

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

is linearly independent, but it is not a basis for $\mathbb{R}^3$ because it does not span $\mathbb{R}^3$.

(iii)

If

$$H = \text{Span}(\mathbf{b}_1, \dots, \mathbf{b}_p),$$

then

$$\{\mathbf{b}_1, \dots, \mathbf{b}_p\}$$

is a basis for H.

Answer

$$\boxed{\text{False.}}$$

The set

$$\{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$$

spans H, but it may not be linearly independent.

A basis must be a spanning set that is also linearly independent.

For example,

$$\text{Span}\left(\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}2\\0\end{bmatrix}\right)$$

is the same as

$$\text{Span}\left(\begin{bmatrix}1\\0\end{bmatrix}\right),$$

but the two-vector set is linearly dependent.

(iv)

The columns of an invertible $n \times n$ matrix form a basis for $\mathbb{R}^n$.

Answer

$$\boxed{\text{True.}}$$

If an $n \times n$ matrix is invertible, then its columns are linearly independent. Since there are n linearly independent vectors in $\mathbb{R}^n$, they automatically form a basis for $\mathbb{R}^n$.

(v)

A basis is a linearly independent set that is as large as possible.

Answer

$$\boxed{\text{True.}}$$

A basis can be described as a maximal linearly independent set. This means that once a set is a basis, we cannot add another vector from the same space without making the set linearly dependent.

(vi)

A basis is a spanning set that is as large as possible.

Answer

$$\boxed{\text{False.}}$$

A basis is not a spanning set that is as large as possible. Instead, a basis is a spanning set that is as small as possible while still spanning the space.

For example, in $\mathbb{R}^3$, a basis has exactly 3 vectors. If we add more vectors, the set may still span $\mathbb{R}^3$, but it will become linearly dependent.

(vii)

Row operations preserve the linear dependence relations among the rows of A.

Answer

$$\boxed{\text{True.}}$$

Elementary row operations are reversible. They change the rows by replacing them with linear combinations of the original rows, but they do not change the row space of the matrix.

Therefore, row operations preserve the dimension of the row space and preserve whether the rows are linearly independent or linearly dependent.

(viii)

If B is an echelon form of a matrix A, the pivot columns of B form a basis for $\text{Col}(A)$.

Answer

$$\boxed{\text{False.}}$$

The pivot columns of B do not usually form a basis for $\text{Col}(A)$.

Instead, we use the pivot column positions of B, then take the corresponding columns from the original matrix A.

Therefore, the correct statement is:

$$\boxed{\text{The pivot columns of } A \text{ form a basis for } \text{Col}(A).}$$

More precisely, after row-reducing A to an echelon form B, we identify the pivot columns in B. Then the corresponding columns of the original matrix A form a basis for $\text{Col}(A)$.

Final Answers Summary

1. A basis for the symmetric 2×2 matrices is

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

2. (a) Yes, it is a basis.
   (b) No, it contains the zero vector.
   (c) No, it has four vectors in $\mathbb{R}^3$.
   (d) No, it has only two vectors in $\mathbb{R}^3$.

3.

$$\text{Nul}(A) : \left\{ \begin{bmatrix} -12 \\ -5 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -10 \\ -3 \\ 0 \\ 2 \end{bmatrix} \right\}.$$

$$\text{Col}(A) : \left\{ \begin{bmatrix} -2 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 4 \\ -6 \\ 8 \end{bmatrix} \right\}.$$

$$\text{Row}(A) : \left\{ \begin{bmatrix} 1 & 0 & 6 & 5 \end{bmatrix}, \begin{bmatrix} 0 & 1 & \frac{5}{2} & \frac{3}{2} \end{bmatrix} \right\}.$$

4. A basis for the plane $x + 2y + z = 0$ is

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

5.

Statement	Answer
(*i*)	False
(*ii*)	False
(*iii*)	False
(*iv*)	True
(*v*)	True
(*vi*)	False
(*vii*)	True
(*viii*)	False

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