Sets and Relations Homework 2
Detailed Solutions

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Problem 1

Let
$$A = \{0, 1, 2, 3\}.$$
For each of the following relations, determine whether or not it is an equivalence relation. If it is an equivalence relation, give the corresponding partition of $\{0, 1, 2, 3\}$ into equivalence classes.

Important Definitions

A relation R on a set A is an **equivalence relation** if it satisfies all three of the following properties:

- **Reflexive:** For every $a \in A$,
$$(a, a) \in R.$$
- **Symmetric:** If $(a, b) \in R$, then
$$(b, a) \in R.$$
- **Transitive:** If $(a, b) \in R$ and $(b, c) \in R$, then
$$(a, c) \in R.$$

If a relation is an equivalence relation, it separates the set into groups called **equivalence classes**. These equivalence classes form a **partition** of the set.

Problem 1(a)

The relation is
$$R = \{(0, 0), (1, 1), (2, 2), (3, 3)\}.$$

Reflexive

Since

$$(0,0),(1,1),(2,2),(3,3) \in R,$$

every element of A is related to itself.

Therefore, R is reflexive.

Symmetric

The only pairs in R are pairs of the form (a,a).

Since reversing (a,a) still gives (a,a), every pair has its reverse in R.

Therefore, R is symmetric.

Transitive

If $(a,b) \in R$ and $(b,c) \in R$, then because the only pairs in R are self-pairs, we must have

$$a = b = c.$$

Thus,

$$(a,c) = (a,a) \in R.$$

Therefore, R is transitive.

Conclusion

The relation is reflexive, symmetric, and transitive.

Therefore,

$$\boxed{R \text{ is an equivalence relation.}}$$

Each element is only related to itself, so the equivalence classes are

$$[0] = \{0\}, \qquad [1] = \{1\}, \qquad [2] = \{2\}, \qquad [3] = \{3\}.$$

Thus, the corresponding partition is

$$\boxed{\{\{0\},\{1\},\{2\},\{3\}\}.}$$

Problem 1(b)

The relation is

$$R = \{(0,0),(0,2),(0,3),(2,0),(2,2),(2,3),(3,2),(3,3)\}.$$

Reflexive

For R to be reflexive on

$$A = \{0,1,2,3\},$$

we need all of the following pairs:

$$(0,0),(1,1),(2,2),(3,3).$$

We see that

$$(0,0),(2,2),(3,3) \in R,$$

but

$$(1,1) \notin R.$$

Therefore, R is not reflexive.

$$\boxed{\text{Not reflexive.}}$$

Since an equivalence relation must be reflexive, symmetric, and transitive, this is enough to conclude that R is not an equivalence relation.

$$\boxed{R \text{ is not an equivalence relation.}}$$

Additional Note

We can also see that this relation has another issue: it is not symmetric.

For example,

$$(0,3) \in R,$$

but

$$(3,0) \notin R.$$

So R also fails symmetry.

Problem 1(c)

The relation is

$$R = \{(0,0),(1,1),(1,2),(2,1),(2,2),(3,3)\}.$$

Reflexive

For R to be reflexive, we need

$$(0,0),(1,1),(2,2),(3,3).$$

All four of these pairs are in R.
Therefore, R is reflexive.

Symmetric

We check whether every pair has its reverse.
The pair

$$(1,2) \in R$$

and its reverse

$$(2,1) \in R.$$

The other pairs are self-pairs:

$$(0,0),(1,1),(2,2),(3,3),$$

and these automatically satisfy symmetry.
Therefore, R is symmetric.

Transitive

The only nontrivial connections are between 1 and 2.
We have

$$1R2$$

and

$$2R1.$$

Then transitivity requires

$$1R1$$

and

$$2R2,$$

which are both already in R.
Also, since 0 is only related to itself and 3 is only related to itself, there are no transitivity problems involving 0 or 3.
Therefore, R is transitive.

Conclusion

The relation is reflexive, symmetric, and transitive.

Therefore,

$$\boxed{R \text{ is an equivalence relation.}}$$

The equivalence classes are

$$[0] = \{0\},$$

$$[1] = \{1, 2\},$$

$$[2] = \{1, 2\},$$

and

$$[3] = \{3\}.$$

Thus, the corresponding partition is

$$\boxed{\{\{0\}, \{1, 2\}, \{3\}\}.}$$

Problem 1(d)

The relation is

$$R = \{(0,0),(0,1),(0,2),(0,3),\\ (1,0),(1,1),(1,2),(1,3),\\ (2,0),(2,1),(2,2),(2,3),\\ (3,0),(3,1),(3,2),(3,3)\}.$$

This relation says that everything is related to everything.

Equivalently,

$$R = A \times A.$$

Reflexive

Since $R = A \times A$, every possible ordered pair is in R.

In particular,

$$(0,0),(1,1),(2,2),(3,3) \in R.$$

Therefore, R is reflexive.

Symmetric

If $(a,b) \in R$, then since R contains every possible ordered pair, we also have

$$(b,a) \in R.$$

Therefore, R is symmetric.

Transitive

If $(a,b) \in R$ and $(b,c) \in R$, then since R contains every possible ordered pair, we automatically have

$$(a,c) \in R.$$

Therefore, R is transitive.

Conclusion

The relation is reflexive, symmetric, and transitive.

Therefore,

$$\boxed{R \text{ is an equivalence relation.}}$$

Since every element is related to every other element, there is only one equivalence class:

$$[0] = [1] = [2] = [3] = \{0,1,2,3\}.$$

Thus, the corresponding partition is

$$\boxed{\{\{0,1,2,3\}\}.}$$

Problem 2

Let A be the set of all polynomials. Suppose that two polynomials are related if they have the same roots, including multiplicities.

Show that this is an equivalence relation. What is the equivalence class of

$$x^2 - 4?$$

Solution

Define a relation R on the set of polynomials by saying

$$pRq$$

if and only if p and q have the same roots, including multiplicities.

We need to show that R is reflexive, symmetric, and transitive.

Reflexive

Every polynomial has the same roots as itself.

Therefore, for every polynomial p,

$$pRp.$$

So R is reflexive.

$$\boxed{\text{Reflexive: Yes}}$$

Symmetric

Suppose

$$pRq.$$

This means p and q have the same roots with the same multiplicities.

But then q and p also have the same roots with the same multiplicities.

Therefore,

$$qRp.$$

So R is symmetric.

$$\boxed{\text{Symmetric: Yes}}$$

Transitive

Suppose

$$pRq$$

and

$$qRr.$$

The statement pRq means p and q have the same roots with the same multiplicities.
The statement qRr means q and r have the same roots with the same multiplicities.
Therefore, p and r must also have the same roots with the same multiplicities.
Thus,

$$pRr.$$

So R is transitive.

$$\boxed{\text{Transitive: Yes}}$$

Conclusion

Since R is reflexive, symmetric, and transitive,

$$\boxed{R \text{ is an equivalence relation.}}$$

Equivalence Class of $x^2 - 4$

We factor:

$$x^2 - 4 = (x - 2)(x + 2).$$

So the roots of $x^2 - 4$ are

$$x = 2$$

and

$$x = -2.$$

Each root has multiplicity 1.

Therefore, the equivalence class of $x^2 - 4$ is the set of all polynomials with exactly the same roots, including multiplicities.

So the equivalence class is

$$[x^2 - 4] = \{p(x) : p(x) \text{ has roots 2 and } -2, \text{ each with multiplicity } 1\}.$$

Equivalently,

$$\boxed{[x^2 - 4] = \{c(x^2 - 4) : c \neq 0\}.}$$

Here c is a nonzero constant.

Problem 3

Consider the partition

$$\{0\}, \qquad \{1,4\}, \qquad \{2,3,5\}.$$

This corresponds to an equivalence relation R on

$$\{0,1,2,3,4,5\}.$$

List all pairs (x,y) such that xRy.

Solution

A partition determines an equivalence relation by saying:

$$xRy \quad \text{if and only if} \quad x \text{ and } y \text{ are in the same block of the partition.}$$

The partition is

$$\{0\}, \qquad \{1,4\}, \qquad \{2,3,5\}.$$

So we list all ordered pairs from each block.

Block 1: $\{0\}$

The only ordered pair is

$$(0,0).$$

Block 2: $\{1,4\}$

The ordered pairs are

$$(1,1),(1,4),(4,1),(4,4).$$

Block 3: $\{2,3,5\}$

The ordered pairs are

$$\begin{gathered}(2,2),(2,3),(2,5),\\(3,2),(3,3),(3,5),\\(5,2),(5,3),(5,5).\end{gathered}$$

Therefore,

$$\boxed{\begin{aligned}R=\{&(0,0),\\&(1,1),(1,4),(4,1),(4,4),\\&(2,2),(2,3),(2,5),\\&(3,2),(3,3),(3,5),\\&(5,2),(5,3),(5,5)\}.\end{aligned}}$$

Problem 4

Let A be the collection of all ordered pairs (x, y) of integers with

$$y \neq 0.$$

Define a function

$$f : A \to \mathbb{R}$$

by

$$f(x, y) = \frac{x}{y}.$$

What is the codomain of this function, and what is its range? Determine whether this function is injective and whether it is surjective.

Solution

The domain is

$$A = \{(x, y) : x, y \in \mathbb{Z}, y \neq 0\}.$$

The function is defined by

$$f(x, y) = \frac{x}{y}.$$

Codomain

The problem explicitly says

$$f : A \to \mathbb{R}.$$

Therefore, the codomain is

$$\boxed{\mathbb{R}.}$$

Range

The range is the set of all possible outputs of the function.

Since x and y are integers and $y \neq 0$, every output has the form

$$\frac{x}{y},$$

where $x, y \in \mathbb{Z}$ and $y \neq 0$.

This is exactly the set of rational numbers:

$$\mathbb{Q}.$$

Therefore, the range is

$$\boxed{\mathbb{Q}.}$$

Injective

A function is injective if different inputs always give different outputs.

In other words, f is injective if

$$f(a) = f(b)$$

implies

$$a = b.$$

Here, the function is not injective because different ordered pairs can give the same fraction.

For example,

$$f(1,2) = \frac{1}{2}$$

and

$$f(2,4) = \frac{2}{4} = \frac{1}{2}.$$

But

$$(1,2) \neq (2,4).$$

Therefore, f is not injective.

$$\boxed{f \text{ is not injective.}}$$

Surjective

A function

$$f : A \to \mathbb{R}$$

is surjective if every real number is hit by the function.

That means for every real number $r \in \mathbb{R}$, there must be some pair $(x, y) \in A$ such that

$$f(x, y) = r.$$

But the range of f is only

$$\mathbb{Q},$$

the rational numbers.

There are real numbers that are not rational, such as

$$\sqrt{2}$$

and

$$\pi.$$

There is no pair of integers (x, y) with $y \neq 0$ such that

$$\frac{x}{y} = \sqrt{2}.$$

Therefore, f is not surjective onto $\mathbb{R}$.

$$\boxed{f \text{ is not surjective.}}$$

Final Answer

$$\boxed{\text{Codomain} = \mathbb{R}}$$

$$\boxed{\text{Range} = \mathbb{Q}}$$

$$\boxed{f \text{ is not injective and not surjective.}}$$

Problem 5

Give an example of two functions f and g from $\mathbb{R}$ to $\mathbb{R}$ such that:

- f is surjective but not injective,
- g is injective but not surjective.

Solution

We need functions

$$f : \mathbb{R} \to \mathbb{R}$$

and

$$g : \mathbb{R} \to \mathbb{R}.$$

Example of f: Surjective but Not Injective

Let

$$f(x) = x^3 - x.$$

Why f is not injective

We compute:

$$f(-1) = (-1)^3 - (-1) = -1 + 1 = 0,$$
$$f(0) = 0^3 - 0 = 0,$$

and

$$f(1) = 1^3 - 1 = 1 - 1 = 0.$$

So

$$f(-1) = f(0) = f(1) = 0.$$

But

$$-1 \neq 0 \neq 1.$$

Therefore, different inputs can give the same output.
Thus,

$$\boxed{f \text{ is not injective.}}$$

Why f is surjective

The function

$$f(x) = x^3 - x$$

is a cubic polynomial.
As $x \to \infty$,

$$x^3 - x \to \infty.$$

As $x \to -\infty$,

$$x^3 - x \to -\infty.$$

Since f is continuous and its values go from negative infinity to positive infinity, it hits every real number.

Therefore,

$$\boxed{f \text{ is surjective.}}$$

Example of g: Injective but Not Surjective

Let

$$g(x) = e^x.$$

Why g is injective

The function

$$g(x) = e^x$$

is strictly increasing.

That means if

$$x_1 < x_2,$$

then

$$e^{x_1} < e^{x_2}.$$

Therefore, two different inputs cannot give the same output.

Thus,

$$\boxed{g \text{ is injective.}}$$

Why g is not surjective

The codomain is $\mathbb{R}$, but the outputs of e^x are always positive.

That is,

$$e^x > 0$$

for every real number x.

So the range is

$$(0, \infty).$$

This means negative numbers and 0 are never hit.

For example, there is no real number x such that

$$e^x = -1.$$

Therefore,

$$\boxed{g \text{ is not surjective.}}$$

Final Answer

One possible answer is

$$\boxed{f(x) = x^3 - x}$$

and

$$\boxed{g(x) = e^x.}$$

Here,

$$f : \mathbb{R} \to \mathbb{R}$$

is surjective but not injective, while

$$g : \mathbb{R} \to \mathbb{R}$$

is injective but not surjective.

Problem 6

Let $\lfloor x \rfloor$ be the floor function, and define

$$g(x) = \lfloor x + 0.5 \rfloor.$$

Problem 6(a)

Compute

$$g(5), \qquad g(1.3), \qquad g(2.5), \qquad g(-1.7).$$

Solution

Recall that $\lfloor x \rfloor$ means the greatest integer less than or equal to x.

Compute $g(5)$

$$g(5) = \lfloor 5 + 0.5 \rfloor.$$

So

$$g(5) = \lfloor 5.5 \rfloor.$$

The greatest integer less than or equal to 5.5 is 5.
Thus,

$$\boxed{g(5) = 5.}$$

Compute $g(1.3)$

$$g(1.3) = \lfloor 1.3 + 0.5 \rfloor.$$

So

$$g(1.3) = \lfloor 1.8 \rfloor.$$

The greatest integer less than or equal to 1.8 is 1.
Thus,

$$\boxed{g(1.3) = 1.}$$

Compute $g(2.5)$

$$g(2.5) = \lfloor 2.5 + 0.5 \rfloor.$$

So

$$g(2.5) = \lfloor 3.0 \rfloor.$$

Since

$$\lfloor 3.0 \rfloor = 3,$$

we get

$$\boxed{g(2.5) = 3.}$$

Compute $g(-1.7)$

$$g(-1.7) = \lfloor -1.7 + 0.5 \rfloor.$$

So

$$g(-1.7) = \lfloor -1.2 \rfloor.$$

The greatest integer less than or equal to -1.2 is -2.
Thus,

$$\boxed{g(-1.7) = -2.}$$

Final Answer for 6(a)

$$\boxed{g(5) = 5}$$

$$\boxed{g(1.3) = 1}$$

$$\boxed{g(2.5) = 3}$$

$$\boxed{g(-1.7) = -2}$$

Problem 6(b)

Recall that the floor function satisfies

$$x - 1 < \lfloor x \rfloor \le x.$$

Using this, write down an inequality that looks like

$$\_\_\_\_\_\_\_\_\_\_ < g(x) \le \_\_\_\_\_\_\_\_\_\_.$$

Solution

We are given

$$g(x) = \lfloor x + 0.5 \rfloor.$$

The floor inequality says that for any real number t,

$$t - 1 < \lfloor t \rfloor \le t.$$

Here, let

$$t = x + 0.5.$$

Then

$$t - 1 = (x + 0.5) - 1 = x - 0.5.$$

Also,

$$\lfloor t \rfloor = \lfloor x + 0.5 \rfloor = g(x).$$

So the inequality becomes

$$x - 0.5 < g(x) \le x + 0.5.$$

Therefore,

$$\boxed{x - 0.5 < g(x) \le x + 0.5.}$$

Equivalently,

$$\boxed{x - \frac{1}{2} < g(x) \le x + \frac{1}{2}.}$$

Problem 6(c)

Since $g(x)$ is a floor function, it is always picking an integer. Why is it always picking the integer closest to x, or at least one tied for closest?

Solution

From part (b), we proved that

$$x - \frac{1}{2} < g(x) \le x + \frac{1}{2}.$$

This means that $g(x)$ is an integer that lies within half a unit of x.

In other words,

$$-\frac{1}{2} < g(x) - x \le \frac{1}{2}.$$

Therefore,

$$|g(x) - x| \le \frac{1}{2}.$$

So $g(x)$ is always an integer whose distance from x is at most $\frac{1}{2}$.

Now, on the number line, the closest integer to any real number must be within distance $\frac{1}{2}$ of that number.

The function

$$g(x) = \lfloor x + 0.5 \rfloor$$

chooses that integer.

If x is exactly halfway between two integers, then there is a tie. For example,

$$x = 2.5.$$

The two closest integers are 2 and 3, and both are distance 0.5 from 2.5.

In this case,

$$g(2.5) = \lfloor 3.0 \rfloor = 3.$$

So $g(x)$ chooses one of the tied closest integers.

Therefore,

$$\boxed{g(x) = \lfloor x + 0.5 \rfloor \text{ rounds } x \text{ to the nearest integer.}}$$

If there is a tie, it rounds up to the larger integer.

$$\boxed{\text{Thus, } g(x) \text{ always picks the integer closest to } x\text{, or one tied for closest.}}$$

Summary of Main Answers

Problem 1

Part	Equivalence Relation?	Partition if Equivalence Relation
(a)	Yes	$\{\{0\}, \{1\}, \{2\}, \{3\}\}$
(b)	No	Not applicable
(c)	Yes	$\{\{0\}, \{1, 2\}, \{3\}\}$
(d)	Yes	$\{\{0, 1, 2, 3\}\}$

Problem 2

The relation "having the same roots including multiplicities" is an equivalence relation.

$$\boxed{[x^2 - 4] = \{c(x^2 - 4) : c \neq 0\}.}$$

Problem 3

$$\boxed{\begin{aligned} R = \{&(0,0), \\ &(1,1), (1,4), (4,1), (4,4), \\ &(2,2), (2,3), (2,5), \\ &(3,2), (3,3), (3,5), \\ &(5,2), (5,3), (5,5)\}. \end{aligned}}$$

Problem 4

$$\boxed{\text{Codomain} = \mathbb{R}}$$

$$\boxed{\text{Range} = \mathbb{Q}}$$

$$\boxed{f \text{ is not injective and not surjective.}}$$

Problem 5

One possible answer is

$$\boxed{f(x) = x^3 - x}$$

and

$$\boxed{g(x) = e^x.}$$

Problem 6

$$\boxed{g(5) = 5}$$

$$\boxed{g(1.3) = 1}$$

$$\boxed{g(2.5) = 3}$$

$$\boxed{g(-1.7) = -2}$$

$$\boxed{x - \frac{1}{2} < g(x) \leq x + \frac{1}{2}}$$

$$\boxed{g(x) = \lfloor x + 0.5 \rfloor \text{ rounds } x \text{ to the nearest integer, rounding ties upward.}}$$