Sets and Relations Homework 1
Detailed Solutions

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Problem 1

List out all of the elements of

$$\{x \mid x \text{ is an integer and } -4 < x < 3\}.$$

Solution

We are looking for all integers strictly between -4 and 3.

The inequality

$$-4 < x < 3$$

means that x must be larger than -4 and smaller than 3.

The integers between -4 and 3 are

$$-3, -2, -1, 0, 1, 2.$$

Therefore,

$$\boxed{\{x \mid x \text{ is an integer and } -4 < x < 3\} = \{-3, -2, -1, 0, 1, 2\}.}$$

Problem 2

Let A be the real numbers, and let

$$B = \{-1, 1\}.$$

If you were to draw $A \times B$ in the xy-plane, what would it look like?

Solution

The Cartesian product $A \times B$ is defined as

$$A \times B = \{(a, b) \mid a \in A, b \in B\}.$$

Since A is the set of real numbers, the first coordinate can be any real number:

$$a \in \mathbb{R}.$$

Since

$$B = \{-1, 1\},$$

the second coordinate can only be -1 or 1.

Thus,

$$A \times B = \{(x, y) \mid x \in \mathbb{R}, y \in \{-1, 1\}\}.$$

This means the set consists of all points whose y-coordinate is either 1 or -1.

So in the xy-plane, $A \times B$ looks like two horizontal lines:

$$y = 1$$

and

$$y = -1.$$

Therefore,

$$\boxed{A \times B \text{ is two horizontal lines: one at } y = 1 \text{ and one at } y = -1.}$$

Problem 3

For each of the following relations, determine whether R is reflexive, symmetric, antisymmetric, and transitive.

Definitions

Let R be a relation on a set A.

- R is **reflexive** if for every $x \in A$,
$$xRx.$$
- R is **symmetric** if whenever xRy, then
$$yRx.$$
- R is **antisymmetric** if whenever xRy and yRx, then
$$x = y.$$
- R is **transitive** if whenever xRy and yRz, then
$$xRz.$$

Problem 3(a)

Let
$$A = \{1, 2, 3\},$$
and
$$R = \{(1,1), (1,2), (2,2), (2,3), (3,2), (3,3)\}.$$

Reflexive

For R to be reflexive, we need
$$(1,1), (2,2), (3,3)$$
to all be in R.

Checking the relation:
$$(1,1) \in R, \quad (2,2) \in R, \quad (3,3) \in R.$$

Therefore, R is reflexive.

$$\boxed{\text{Reflexive: Yes}}$$

Symmetric

For R to be symmetric, whenever $(a, b) \in R$, we must also have $(b, a) \in R$.

Now,

$$(1, 2) \in R.$$

If R were symmetric, then we would need

$$(2, 1) \in R.$$

But

$$(2, 1) \notin R.$$

Therefore, R is not symmetric.

$$\boxed{\text{Symmetric: No}}$$

Antisymmetric

For R to be antisymmetric, if both (a, b) and (b, a) are in R, then we must have $a = b$.

But we have

$$(2, 3) \in R$$

and

$$(3, 2) \in R.$$

Since

$$2 \neq 3,$$

this violates antisymmetry.

Therefore, R is not antisymmetric.

$$\boxed{\text{Antisymmetric: No}}$$

Transitive

For R to be transitive, whenever $(a, b) \in R$ and $(b, c) \in R$, we must also have $(a, c) \in R$.

We check possible chains.

Since

$$(1, 2) \in R$$

and

$$(2, 3) \in R,$$

transitivity would require

$$(1, 3) \in R.$$

But

$$(1, 3) \notin R.$$

Therefore, R is not transitive.

$$\boxed{\text{Transitive: No}}$$

Final Answer for 3(a)

Property	Answer
Reflexive	Yes
Symmetric	No
Antisymmetric	No
Transitive	No

Problem 3(b)

Let $A = \mathbb{R}$. Define a relation R by

$$xRy \quad \text{if and only if} \quad x = 1 \text{ or } y = 1.$$

Reflexive

For R to be reflexive, we need

$$xRx$$

for every real number x.

Using the definition of R,

$$xRx$$

means

$$x = 1 \text{ or } x = 1.$$

This is only true when $x = 1$.

For example, if $x = 2$, then

$$2R2$$

would require $2 = 1$ or $2 = 1$, which is false.

Therefore, R is not reflexive.

$$\boxed{\text{Reflexive: No}}$$

Symmetric

Suppose

$$xRy.$$

Then by definition,

$$x = 1 \text{ or } y = 1.$$

To check symmetry, we need to see whether

$$yRx$$

is true.

Now yRx means

$$y = 1 \text{ or } x = 1.$$

But this is the same condition as

$$x = 1 \text{ or } y = 1.$$

Therefore, whenever xRy is true, yRx is also true.

So R is symmetric.

$$\boxed{\text{Symmetric: Yes}}$$

Antisymmetric

For antisymmetry, if both

$$xRy$$

and

$$yRx$$

are true, then we must have $x = y$.

Consider $x = 1$ and $y = 2$.

Then

$$1R2$$

is true because $x = 1$.

Also,

$$2R1$$

is true because $y = 1$.

But

$$1 \neq 2.$$

Therefore, R is not antisymmetric.

$$\boxed{\text{Antisymmetric: No}}$$

Transitive

For transitivity, if

$$xRy$$

and

$$yRz,$$

then we must have

$$xRz.$$

We show this fails by counterexample.

Let

$$x = 2, \quad y = 1, \quad z = 3.$$

Then

$$2R1$$

is true because $y = 1$.

Also,

$$1R3$$

is true because $y = 1$ in the pair $(1, 3)$, meaning the first coordinate is 1.

However,

$$2R3$$

would require

$$2 = 1 \text{ or } 3 = 1,$$

which is false.

Thus,

$$2R1 \quad \text{and} \quad 1R3$$

are true, but

$$2R3$$

is false.

Therefore, R is not transitive.

$$\boxed{\text{Transitive: No}}$$

Final Answer for 3(b)

Property	Answer
Reflexive	No
Symmetric	Yes
Antisymmetric	No
Transitive	No

Problem 3(c)

Let A be the set of all lines in the plane. Define

$$aRb$$

if line a is perpendicular to line b.

Reflexive

For R to be reflexive, every line would have to be perpendicular to itself.

But a line is not perpendicular to itself.

Therefore, R is not reflexive.

$$\boxed{\text{Reflexive: No}}$$

Symmetric

If line a is perpendicular to line b, then line b is also perpendicular to line a.

Therefore, R is symmetric.

$$\boxed{\text{Symmetric: Yes}}$$

Antisymmetric

For antisymmetry, if

$$aRb$$

and

$$bRa,$$

then we would need

$$a = b.$$

But perpendicular lines are usually different lines.

For example, the x-axis is perpendicular to the y-axis, and the y-axis is perpendicular to the x-axis. However, they are not the same line.

Therefore, R is not antisymmetric.

$$\boxed{\text{Antisymmetric: No}}$$

Transitive

For transitivity, if line a is perpendicular to line b, and line b is perpendicular to line c, then line a would have to be perpendicular to line c.

But this is not always true.

For example, let

$$a = \text{the } x\text{-axis},$$

$$b = \text{the } y\text{-axis},$$

and

$$c = \text{another horizontal line.}$$

Then a is perpendicular to b, and b is perpendicular to c.
However, a and c are both horizontal, so they are parallel, not perpendicular.
Therefore, R is not transitive.

$$\boxed{\text{Transitive: No}}$$

Final Answer for 3(c)

Property	Answer
Reflexive	No
Symmetric	Yes
Antisymmetric	No
Transitive	No

Problem 4

Repeat Problem 3 for the following relations.

Problem 4(a)

Let A be the set of all students at UCR. Define

$$xRy$$

if there is at least one class where x and y are both enrolled.

Reflexive

For R to be reflexive, every student x must satisfy

$$xRx.$$

This means student x shares at least one class with themselves.

Assuming every student at UCR is enrolled in at least one class, each student is in the same class as themselves.

Therefore, under this usual assumption, R is reflexive.

$$\boxed{\text{Reflexive: Yes}}$$

Symmetric

If student x shares a class with student y, then student y also shares that same class with student x.

Therefore, R is symmetric.

$$\boxed{\text{Symmetric: Yes}}$$

Antisymmetric

For antisymmetry, if

$$xRy$$

and

$$yRx,$$

then we would need

$$x = y.$$

But two different students can be in the same class.

For example, two different students may both be enrolled in Math 11.

Therefore, R is not antisymmetric.

$$\boxed{\text{Antisymmetric: No}}$$

Transitive

For transitivity, if student x shares a class with student y, and student y shares a class with student z, then student x would have to share a class with student z.

This is not always true.

For example, suppose:

$$x \text{ and } y \text{ are both in Math 11,}$$

and

$$y \text{ and } z \text{ are both in CS 11.}$$

It does not necessarily follow that x and z share a class.

Therefore, R is not transitive.

$$\boxed{\text{Transitive: No}}$$

Final Answer for 4(a)

Property	Answer
Reflexive	Yes
Symmetric	Yes
Antisymmetric	No
Transitive	No

Problem 4(b)

Let A be the set of all US states. Define

$$aRb$$

if either

$$a = b$$

or a and b share a land border.

Reflexive

For every state a, we have

$$a = a.$$

Since the relation says aRb if either $a = b$ or they share a land border, every state is related to itself because $a = a$.

Therefore, R is reflexive.

$$\boxed{\text{Reflexive: Yes}}$$

Symmetric

If state a shares a land border with state b, then state b also shares a land border with state a.

Also, if $a = b$, then $b = a$.

Therefore, R is symmetric.

$$\boxed{\text{Symmetric: Yes}}$$

Antisymmetric

For antisymmetry, if

$$aRb$$

and

$$bRa,$$

then we would need

$$a = b.$$

But two different states can share a land border.

For example, California and Arizona share a land border, so

$$\text{California } R \text{ Arizona}$$

and

$$\text{Arizona } R \text{ California}.$$

But California and Arizona are not the same state.
Therefore, R is not antisymmetric.

$$\boxed{\text{Antisymmetric: No}}$$

Transitive

For transitivity, if state a is related to state b, and state b is related to state c, then state a must be related to state c.

This is not always true.

For example:

$$\text{California shares a land border with Oregon,}$$

and

$$\text{Oregon shares a land border with Washington.}$$

But California and Washington do not share a land border.
Therefore, R is not transitive.

$$\boxed{\text{Transitive: No}}$$

Final Answer for 4(b)

Property	Answer
Reflexive	Yes
Symmetric	Yes
Antisymmetric	No
Transitive	No

Problem 4(c)

Let A be the set of all people in Riverside. Define

$$xRy$$

if x and y share the same first name.

Reflexive

Every person has the same first name as themselves.

Therefore,

$$xRx$$

for every person x in Riverside.

So R is reflexive.

$$\boxed{\text{Reflexive: Yes}}$$

Symmetric

If person x has the same first name as person y, then person y also has the same first name as person x.

Therefore, R is symmetric.

$$\boxed{\text{Symmetric: Yes}}$$

Antisymmetric

For antisymmetry, if

$$xRy$$

and

$$yRx,$$

then we would need

$$x = y.$$

But two different people can have the same first name.

For example, two different people in Riverside may both be named Alex.

Thus, xRy and yRx can both be true while $x \neq y$.

Therefore, R is not antisymmetric.

$$\boxed{\text{Antisymmetric: No}}$$

Transitive

For transitivity, if

$$xRy$$

and

$$yRz,$$

then we need

$$xRz.$$

If x has the same first name as y, and y has the same first name as z, then x and z must also have the same first name.

Therefore, R is transitive.

$$\boxed{\text{Transitive: Yes}}$$

Final Answer for 4(c)

Property	Answer
Reflexive	Yes
Symmetric	Yes
Antisymmetric	No
Transitive	Yes

Problem 5

The following claim is not actually true.

Claim

Any relation that is both symmetric and transitive is also reflexive.

The attempted proof says:

> Take any x in A, and choose a y such that xRy. Since R is symmetric, we must also have yRx. Applying transitivity to xRy and yRx, we get xRx. This is true for any x, so the relation is reflexive.

Where did the proof go wrong?

Solution

The mistake happens in the sentence:

> "Take any x in A, and choose a y such that xRy."

This assumes that for every element $x \in A$, there exists some element $y \in A$ such that

$$xRy.$$

But this does not have to be true.

A relation can be symmetric and transitive even if some elements are not related to anything at all.

Counterexample

Let

$$A = \{1, 2\}$$

and let

$$R = \{(1, 1)\}.$$

We check the properties.

Symmetric

The only pair in R is

$$(1, 1).$$

Since its reverse is also

$$(1, 1),$$

the relation is symmetric.

Transitive

The only possible chain is

$$1R1 \quad \text{and} \quad 1R1.$$

This gives

$$1R1,$$

which is already in R.

So the relation is transitive.

Not Reflexive

For R to be reflexive on $A = \{1, 2\}$, we need both

$$(1, 1)$$

and

$$(2, 2)$$

to be in R.

But

$$(2, 2) \notin R.$$

Therefore, R is not reflexive.

So this relation is symmetric and transitive, but not reflexive.

$$\boxed{\text{The proof is wrong because it assumes every } x \in A \text{ is related to some } y.}$$

$$\boxed{\text{That assumption is not guaranteed by symmetry or transitivity.}}$$

Summary Table

Problem	Reflexive	Symmetric	Antisymmetric	Transitive
$3(a)$	Yes	No	No	No
$3(b)$	No	Yes	No	No
$3(c)$	No	Yes	No	No
$4(a)$	Yes	Yes	No	No
$4(b)$	Yes	Yes	No	No
$4(c)$	Yes	Yes	No	Yes

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