MATH 031 Applied Linear Algebra
Midterm 2 Review Problems: Detailed Solutions

Khoi Vo

Source Note

These solutions are written for the problems visible in the uploaded review PDF and the additional uploaded page for Section 2.2 Problems 39–43. The complete requested set is included below.

Section 2.2: The Inverse of a Matrix

Problems 1–4

For a 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

if

$$ad - bc \neq 0,$$

then

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Problem 1

$$A = \begin{bmatrix} 8 & 3 \\ 5 & 2 \end{bmatrix}.$$

Compute the determinant:

$$\det(A) = 8(2) - 3(5) = 16 - 15 = 1.$$

Since $\det(A) \neq 0$, the matrix is invertible. Thus,

$$A^{-1} = \frac{1}{1} \begin{bmatrix} 2 & -3 \\ -5 & 8 \end{bmatrix}.$$

Therefore,

$$\boxed{A^{-1} = \begin{bmatrix} 2 & -3 \\ -5 & 8 \end{bmatrix}}$$

Problem 2

$$A = \begin{bmatrix} 3 & 1 \\ 7 & 2 \end{bmatrix}.$$

Compute the determinant:

$$\det(A) = 3(2) - 1(7) = 6 - 7 = -1.$$

Since $\det(A) \neq 0$, the matrix is invertible. Therefore,

$$A^{-1} = \frac{1}{-1} \begin{bmatrix} 2 & -1 \\ -7 & 3 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 7 & -3 \end{bmatrix}.$$

Thus,

$$\boxed{A^{-1} = \begin{bmatrix} -2 & 1 \\ 7 & -3 \end{bmatrix}}$$

Problem 3

$$A = \begin{bmatrix} 8 & 3 \\ -7 & -3 \end{bmatrix}.$$

Compute the determinant:

$$\det(A) = 8(-3) - 3(-7) = -24 + 21 = -3.$$

Since $\det(A) \neq 0$, the matrix is invertible. Therefore,

$$A^{-1} = \frac{1}{-3} \begin{bmatrix} -3 & -3 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -\frac{7}{3} & -\frac{8}{3} \end{bmatrix}.$$

Thus,

$$\boxed{A^{-1} = \begin{bmatrix} 1 & 1 \\ -\frac{7}{3} & -\frac{8}{3} \end{bmatrix}}$$

Problem 4

$$A = \begin{bmatrix} 3 & -2 \\ 7 & -4 \end{bmatrix}.$$

Compute the determinant:

$$\det(A) = 3(-4) - (-2)(7) = -12 + 14 = 2.$$

Since $\det(A) \neq 0$, the matrix is invertible. Therefore,

$$A^{-1} = \frac{1}{2} \begin{bmatrix} -4 & 2 \\ -7 & 3 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -\frac{7}{2} & \frac{3}{2} \end{bmatrix}.$$

Thus,

$$\boxed{A^{-1} = \begin{bmatrix} -2 & 1 \\ -\frac{7}{2} & \frac{3}{2} \end{bmatrix}}$$

Problem 9

We are given

$$A = \begin{bmatrix} 1 & 2 \\ 5 & 12 \end{bmatrix}, \qquad \mathbf{b}_1 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}, \qquad \mathbf{b}_2 = \begin{bmatrix} 1 \\ -5 \end{bmatrix},$$
$$\mathbf{b}_3 = \begin{bmatrix} 2 \\ 6 \end{bmatrix}, \qquad \mathbf{b}_4 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}.$$

First compute A^{-1}. We have

$$\det(A) = 1(12) - 2(5) = 12 - 10 = 2.$$

Therefore,

$$A^{-1} = \frac{1}{2}\begin{bmatrix} 12 & -2 \\ -5 & 1 \end{bmatrix} = \begin{bmatrix} 6 & -1 \\ -\frac{5}{2} & \frac{1}{2} \end{bmatrix}.$$

So

$$\boxed{A^{-1} = \begin{bmatrix} 6 & -1 \\ -\frac{5}{2} & \frac{1}{2} \end{bmatrix}}$$

To solve each equation $A\mathbf{x} = \mathbf{b}_i$, use

$$\mathbf{x} = A^{-1}\mathbf{b}_i.$$

For $\mathbf{b}_1$,

$$\mathbf{x}_1 = A^{-1}\mathbf{b}_1 = \begin{bmatrix} 6 & -1 \\ -\frac{5}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -9 \\ 4 \end{bmatrix}.$$

For $\mathbf{b}_2$,

$$\mathbf{x}_2 = A^{-1}\mathbf{b}_2 = \begin{bmatrix} 6 & -1 \\ -\frac{5}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 \\ -5 \end{bmatrix} = \begin{bmatrix} 11 \\ -5 \end{bmatrix}.$$

For $\mathbf{b}_3$,

$$\mathbf{x}_3 = A^{-1}\mathbf{b}_3 = \begin{bmatrix} 6 & -1 \\ -\frac{5}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 2 \\ 6 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \end{bmatrix}.$$

For $\mathbf{b}_4$,

$$\mathbf{x}_4 = A^{-1}\mathbf{b}_4 = \begin{bmatrix} 6 & -1 \\ -\frac{5}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 13 \\ -5 \end{bmatrix}.$$

Thus,

$$\boxed{\mathbf{x}_1 = \begin{bmatrix} -9 \\ 4 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 11 \\ -5 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 6 \\ -2 \end{bmatrix}, \quad \mathbf{x}_4 = \begin{bmatrix} 13 \\ -5 \end{bmatrix}.}$$

Now solve all four systems at once by row reducing

$$[A \; \mathbf{b}_1 \; \mathbf{b}_2 \; \mathbf{b}_3 \; \mathbf{b}_4] .$$

We begin with

$$\left[\begin{array}{cc|cccc} 1 & 2 & -1 & 1 & 2 & 3 \\ 5 & 12 & 3 & -5 & 6 & 5 \end{array}\right].$$

Use

$$R_2 \leftarrow R_2 - 5R_1.$$

Then

$$\left[\begin{array}{cc|cccc} 1 & 2 & -1 & 1 & 2 & 3 \\ 0 & 2 & 8 & -10 & -4 & -10 \end{array}\right].$$

Next use

$$R_2 \leftarrow \frac{1}{2}R_2.$$

So

$$\left[\begin{array}{cc|cccc} 1 & 2 & -1 & 1 & 2 & 3 \\ 0 & 1 & 4 & -5 & -2 & -5 \end{array}\right].$$

Finally use

$$R_1 \leftarrow R_1 - 2R_2.$$

Thus,

$$\left[\begin{array}{cc|cccc} 1 & 0 & -9 & 11 & 6 & 13 \\ 0 & 1 & 4 & -5 & -2 & -5 \end{array}\right].$$

The solution vectors are the columns on the right side:

$$\boxed{\begin{bmatrix} -9 \\ 4 \end{bmatrix}, \begin{bmatrix} 11 \\ -5 \end{bmatrix}, \begin{bmatrix} 6 \\ -2 \end{bmatrix}, \begin{bmatrix} 13 \\ -5 \end{bmatrix}.}$$

Problems 39–43

In Problems 39–42, we use the row-reduction algorithm for inverses. The main idea is

$$[A \mid I] \sim [I \mid A^{-1}].$$

If the left side cannot be row reduced to the identity matrix, then the matrix is not invertible.

Problem 39

Find the inverse of

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 7 \end{bmatrix},$$

if it exists.

Start with the augmented matrix

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 4 & 7 & 0 & 1 \end{array}\right].$$

Use

$$R_2 \leftarrow R_2 - 4R_1.$$

Then

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -1 & -4 & 1 \end{array}\right].$$

Next use

$$R_2 \leftarrow -R_2.$$

So

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 4 & -1 \end{array}\right].$$

Now eliminate the entry above the second pivot:

$$R_1 \leftarrow R_1 - 2R_2.$$

Thus

$$\left[\begin{array}{cc|cc} 1 & 0 & -7 & 2 \\ 0 & 1 & 4 & -1 \end{array}\right].$$

The left side is now the identity matrix, so the right side is the inverse of A. Therefore,

$$\boxed{A^{-1} = \begin{bmatrix} -7 & 2 \\ 4 & -1 \end{bmatrix}}$$

Problem 40

Find the inverse of

$$A = \begin{bmatrix} 5 & 10 \\ 4 & 7 \end{bmatrix},$$

if it exists.
Start with

$$\left[\begin{array}{cc|cc} 5 & 10 & 1 & 0 \\ 4 & 7 & 0 & 1 \end{array}\right].$$

First scale the first row:

$$R_1 \leftarrow \frac{1}{5} R_1.$$

Then

$$\left[\begin{array}{cc|cc} 1 & 2 & \frac{1}{5} & 0 \\ 4 & 7 & 0 & 1 \end{array}\right].$$

Now eliminate the entry below the first pivot:

$$R_2 \leftarrow R_2 - 4R_1.$$

This gives

$$\left[\begin{array}{cc|cc} 1 & 2 & \frac{1}{5} & 0 \\ 0 & -1 & -\frac{4}{5} & 1 \end{array}\right].$$

Next make the second pivot equal to 1:

$$R_2 \leftarrow -R_2.$$

So

$$\left[\begin{array}{cc|cc} 1 & 2 & \frac{1}{5} & 0 \\ 0 & 1 & \frac{4}{5} & -1 \end{array}\right].$$

Now eliminate the entry above the second pivot:

$$R_1 \leftarrow R_1 - 2R_2.$$

Then

$$\left[\begin{array}{cc|cc} 1 & 0 & -\frac{7}{5} & 2 \\ 0 & 1 & \frac{4}{5} & -1 \end{array}\right].$$

Therefore,

$$\boxed{A^{-1} = \begin{bmatrix} -\frac{7}{5} & 2 \\ \frac{4}{5} & -1 \end{bmatrix}}$$

Problem 41

Find the inverse of

$$A = \begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix},$$

if it exists.
Start with the augmented matrix

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ -3 & 1 & 4 & 0 & 1 & 0 \\ 2 & -3 & 4 & 0 & 0 & 1 \end{array}\right].$$

Use

$$R_2 \leftarrow R_2 + 3R_1.$$

This gives

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 2 & -3 & 4 & 0 & 0 & 1 \end{array}\right].$$

Use

$$R_3 \leftarrow R_3 - 2R_1.$$

Then

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & -3 & 8 & -2 & 0 & 1 \end{array}\right].$$

Now eliminate the entry below the second pivot:

$$R_3 \leftarrow R_3 + 3R_2.$$

So

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & 0 & 2 & 7 & 3 & 1 \end{array}\right].$$

Make the third pivot equal to 1:

$$R_3 \leftarrow \frac{1}{2} R_3.$$

Then

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & 0 & 1 & \frac{7}{2} & \frac{3}{2} & \frac{1}{2} \end{array}\right].$$

Now eliminate the entries above the third pivot.
First use

$$R_2 \leftarrow R_2 + 2R_3.$$

Then

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 10 & 4 & 1 \\ 0 & 0 & 1 & \frac{7}{2} & \frac{3}{2} & \frac{1}{2} \end{array}\right].$$

Next use

$$R_1 \leftarrow R_1 + 2R_3.$$

This gives

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 8 & 3 & 1 \\ 0 & 1 & 0 & 10 & 4 & 1 \\ 0 & 0 & 1 & \frac{7}{2} & \frac{3}{2} & \frac{1}{2} \end{array}\right].$$

The left side is the identity matrix, so the right side is A^{-1}. Therefore,

$$\boxed{A^{-1} = \begin{bmatrix} 8 & 3 & 1 \\ 10 & 4 & 1 \\ \frac{7}{2} & \frac{3}{2} & \frac{1}{2} \end{bmatrix}}$$

Problem 42

Find the inverse of

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 4 & -7 & 3 \\ -2 & 6 & -4 \end{bmatrix},$$

if it exists.
Start with

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 4 & -7 & 3 & 0 & 1 & 0 \\ -2 & 6 & -4 & 0 & 0 & 1 \end{array}\right].$$

Eliminate entries below the first pivot:

$$R_2 \leftarrow R_2 - 4R_1, \qquad R_3 \leftarrow R_3 + 2R_1.$$

This gives

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -4 & 1 & 0 \\ 0 & 2 & -2 & 2 & 0 & 1 \end{array}\right].$$

Now eliminate the entry below the second pivot:

$$R_3 \leftarrow R_3 - 2R_2.$$

Then

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -4 & 1 & 0 \\ 0 & 0 & 0 & 10 & -2 & 1 \end{array}\right].$$

The left side has a zero row, so it is impossible to row reduce the left side to I_3. Therefore, A is not invertible.

$$\boxed{A^{-1} \text{ does not exist.}}$$

Problem 43

We first find the inverses of the two displayed matrices.

The 3×3 case. Let

$$A_3 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

Start with

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right].$$

Use

$$R_2 \leftarrow R_2 - R_1, \qquad R_3 \leftarrow R_3 - R_1.$$

Then

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right].$$

Now use

$$R_3 \leftarrow R_3 - R_2.$$

So

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right].$$

Therefore,

$$\boxed{A_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}}$$

The 4×4 case. Let

$$A_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Use the same idea. Subtract each row from the row below it, working from bottom to top:

$$R_4 \leftarrow R_4 - R_3, \qquad R_3 \leftarrow R_3 - R_2, \qquad R_2 \leftarrow R_2 - R_1.$$

This changes the left side to I_4. The right side becomes

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}.$$

Thus,

$$\boxed{A_4^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}}$$

The general $n \times n$ case. Let A be the $n \times n$ matrix whose entries are

$$a_{ij} = \begin{cases} 1, & i \geq j, \\ 0, & i < j. \end{cases}$$

So A has 1's on and below the main diagonal, and 0's above the main diagonal. Based on the 3×3 and 4×4 cases, we guess that $B = A^{-1}$ is the matrix

$$b_{ij} = \begin{cases} 1, & i = j, \\ -1, & i = j+1, \\ 0, & \text{otherwise.} \end{cases}$$

In other words,

$$B = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix},$$

where the entries directly below the main diagonal are -1.
Now prove that this guess is correct.
First compute AB. The (i, j) entry of AB is

$$(AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

Since column j of B has nonzero entries only in positions j and $j+1$, we get

$$(AB)_{ij} = a_{ij} - a_{i,j+1}$$

for $j < n$. If $j = n$, then

$$(AB)_{in} = a_{in}.$$

For $j < n$, there are three cases:

- If $i < j$, then $a_{ij} = 0$ and $a_{i,j+1} = 0$, so $(AB)_{ij} = 0$.
- If $i = j$, then $a_{ij} = 1$ and $a_{i,j+1} = 0$, so $(AB)_{ij} = 1$.
- If $i > j$, then $a_{ij} = 1$ and $a_{i,j+1} = 1$, so $(AB)_{ij} = 0$.

For $j = n$, we have $a_{in} = 1$ only when $i = n$, and $a_{in} = 0$ otherwise. Therefore,

$$AB = I_n.$$

Now compute BA. For $i = 1$, row 1 of B is the first standard row vector, so row 1 of BA equals row 1 of A, which is

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

For $i > 1$, row i of B has -1 in column $i-1$ and 1 in column i. Therefore,

$$(BA)_{ij} = a_{ij} - a_{i-1,j}.$$

Again, there are three cases:

- If $j > i$, then $a_{ij} = 0$ and $a_{i-1,j} = 0$, so $(BA)_{ij} = 0$.
- If $j = i$, then $a_{ij} = 1$ and $a_{i-1,j} = 0$, so $(BA)_{ij} = 1$.
- If $j < i$, then $a_{ij} = 1$ and $a_{i-1,j} = 1$, so $(BA)_{ij} = 0$.

Therefore,

$$BA = I_n.$$

Since both

$$AB = I_n \qquad \text{and} \qquad BA = I_n,$$

we conclude that

$$\boxed{B = A^{-1}.}$$

Section 2.3: Characterizations of Invertible Matrices

Problems 1–8

Unless otherwise specified, each matrix is square. A square matrix is invertible if and only if its determinant is nonzero. Equivalently, it is invertible if and only if it has a pivot in every column.

Problem 1

$$A = \begin{bmatrix} 5 & 7 \\ -3 & -6 \end{bmatrix}.$$

$$\det(A) = 5(-6) - 7(-3) = -30 + 21 = -9.$$

Since $\det(A) \neq 0$,

$$\boxed{A \text{ is invertible.}}$$

Problem 2

$$A = \begin{bmatrix} -4 & 6 \\ 6 & -9 \end{bmatrix}.$$

$$\det(A) = (-4)(-9) - 6(6) = 36 - 36 = 0.$$

Since $\det(A) = 0$,

$$\boxed{A \text{ is not invertible.}}$$

Problem 3

$$A = \begin{bmatrix} 5 & 0 & 0 \\ -3 & -7 & 0 \\ 8 & 5 & -1 \end{bmatrix}.$$

This is a lower triangular matrix. The determinant of a triangular matrix is the product of its diagonal entries:

$$\det(A) = 5(-7)(-1) = 35.$$

Since $\det(A) \neq 0$,

$$\boxed{A \text{ is invertible.}}$$

Problem 4

$$A = \begin{bmatrix} -7 & 0 & 4 \\ 3 & 0 & -1 \\ 2 & 0 & 9 \end{bmatrix}.$$

The second column is the zero column. Therefore, the columns cannot be linearly independent. Hence the matrix cannot be invertible.

$$\boxed{A \text{ is not invertible.}}$$

Problem 5

$$A = \begin{bmatrix} 0 & 3 & -5 \\ 1 & 0 & 2 \\ -4 & -9 & 7 \end{bmatrix}.$$

Notice that

$$3R_1 + 4R_2 + R_3 = \mathbf{0}.$$

Indeed,

$$3\begin{bmatrix} 0 & 3 & -5 \end{bmatrix} + 4\begin{bmatrix} 1 & 0 & 2 \end{bmatrix} + \begin{bmatrix} -4 & -9 & 7 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}.$$

Thus the rows are linearly dependent, so the matrix is not invertible.

$$\boxed{A \text{ is not invertible.}}$$

Problem 6

$$A = \begin{bmatrix} 1 & -5 & -4 \\ 0 & 3 & 4 \\ -3 & 6 & 0 \end{bmatrix}.$$

Notice that

$$3R_1 + 3R_2 + R_3 = \mathbf{0}.$$

Indeed,

$$3\begin{bmatrix} 1 & -5 & -4 \end{bmatrix} + 3\begin{bmatrix} 0 & 3 & 4 \end{bmatrix} + \begin{bmatrix} -3 & 6 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}.$$

Thus the rows are linearly dependent, so the matrix is not invertible.

$$\boxed{A \text{ is not invertible.}}$$

Problem 7

$$A = \begin{bmatrix} -1 & -3 & 0 & 1 \\ 3 & 5 & 8 & -3 \\ -2 & -6 & 3 & 2 \\ 0 & -1 & 2 & 1 \end{bmatrix}.$$

A determinant computation gives

$$\det(A) = 12.$$

Since $\det(A) \neq 0$,

$$\boxed{A \text{ is invertible.}}$$

Problem 8

$$A = \begin{bmatrix} 1 & 3 & 7 & 4 \\ 0 & 5 & 9 & 6 \\ 0 & 0 & 2 & 8 \\ 0 & 0 & 0 & 10 \end{bmatrix}.$$

This is an upper triangular matrix. Therefore,

$$\det(A) = 1(5)(2)(10) = 100.$$

Since $\det(A) \neq 0$,

$$\boxed{A \text{ is invertible.}}$$

Section 4.1: Vector Spaces and Subspaces

Problem 9

Let

$$H = \left\{ \begin{bmatrix} s \\ 3s \\ 2s \end{bmatrix} : s \in \mathbb{R} \right\}.$$

Factor out the parameter s:

$$\begin{bmatrix} s \\ 3s \\ 2s \end{bmatrix} = s \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}.$$

Therefore,

$$H = \operatorname{Span} \left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \right\}.$$

Thus we may take

$$\boxed{\mathbf{v} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}.}$$

Because H is the span of a vector in $\mathbb{R}^3$, it is a subspace of $\mathbb{R}^3$.

$$\boxed{H \text{ is a subspace of } \mathbb{R}^3.}$$

Problem 10

Let

$$H = \left\{ \begin{bmatrix} 2t \\ 0 \\ t \end{bmatrix} : t \in \mathbb{R} \right\}.$$

Factor out the parameter t:

$$\begin{bmatrix} 2t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}.$$

Therefore,

$$H = \operatorname{Span} \left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Since a span is always a subspace,

$$\boxed{H \text{ is a subspace of } \mathbb{R}^3.}$$

Problem 11

Let

$$W = \left\{ \begin{bmatrix} 5b+2c \\ b \\ c \end{bmatrix} : b, c \in \mathbb{R} \right\}.$$

Separate the parameters b and c:

$$\begin{bmatrix} 5b+2c \\ b \\ c \end{bmatrix} = b \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}.$$

Thus,

$$W = \text{Span} \left\{ \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

So we may choose

$$\boxed{\mathbf{u} = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}, \qquad \mathbf{v} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}.}$$

Since W is a span, it is a subspace of $\mathbb{R}^3$.

$$\boxed{W \text{ is a subspace of } \mathbb{R}^3.}$$

Problem 12

Let

$$W = \left\{ \begin{bmatrix} s+3t \\ s-t \\ 2s-t \\ 4t \end{bmatrix} : s, t \in \mathbb{R} \right\}.$$

Separate the parameters:

$$\begin{bmatrix} s+3t \\ s-t \\ 2s-t \\ 4t \end{bmatrix} = s \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ -1 \\ -1 \\ 4 \end{bmatrix}.$$

Therefore,

$$W = \text{Span}\left\{\begin{bmatrix}1\\1\\2\\0\end{bmatrix}, \begin{bmatrix}3\\-1\\-1\\4\end{bmatrix}\right\}.$$

Since W is a span,

$$\boxed{W \text{ is a subspace of } \mathbb{R}^4.}$$

Problem 13

Let

$$\mathbf{v}_1 = \begin{bmatrix}1\\0\\-1\end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix}2\\1\\3\end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix}4\\2\\6\end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix}3\\1\\2\end{bmatrix}.$$

Part (a)

The set

$$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$$

contains exactly three vectors.
The vector $\mathbf{w}$ is not equal to $\mathbf{v}_1$, $\mathbf{v}_2$, or $\mathbf{v}_3$.
Thus,

$$\boxed{\mathbf{w} \notin \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}.}$$

Part (b)

The span

$$\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$$

contains all linear combinations of the vectors. Since these vectors are not all zero, the span contains infinitely many vectors.

$$\boxed{\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \text{ contains infinitely many vectors.}}$$

Part (c)

We test whether

$$\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3.$$

One solution is

$$c_1 = 1, \qquad c_2 = 1, \qquad c_3 = 0.$$

Indeed,

$$\mathbf{v}_1 + \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} = \mathbf{w}.$$

Therefore,

$$\boxed{\mathbf{w} \in \operatorname{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}.}$$

Problem 14

Let

$$\mathbf{w} = \begin{bmatrix} 8 \\ 4 \\ 7 \end{bmatrix}.$$

We want to know if

$$\mathbf{w} \in \operatorname{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}.$$

This means we need to solve

$$c_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + c_3 \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \\ 7 \end{bmatrix}.$$

The augmented matrix row reduces to a matrix with an inconsistent row:

$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & 8 \\ 0 & 1 & 2 & 4 \\ -1 & 3 & 6 & 7 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right].$$

The row

$$0 = 1$$

is impossible. Therefore the system is inconsistent.
Thus,

$$\boxed{\mathbf{w} \notin \operatorname{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}.}$$

Section 4.2: Null Spaces, Column Spaces, and Linear Transformations

Problems 3–6

Problem 3

$$A = \begin{bmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 4 & -2 \end{bmatrix}.$$

Row reduce:

$$\begin{bmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 4 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -7 & 6 \\ 0 & 1 & 4 & -2 \end{bmatrix}.$$

The equations are

$$x_1 - 7x_3 + 6x_4 = 0, \qquad x_2 + 4x_3 - 2x_4 = 0.$$

Let

$$x_3 = s, \qquad x_4 = t.$$

Then

$$x_1 = 7s - 6t, \qquad x_2 = -4s + 2t.$$

Thus,

$$\mathbf{x} = s \begin{bmatrix} 7 \\ -4 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -6 \\ 2 \\ 0 \\ 1 \end{bmatrix}.$$

Therefore,

$$\boxed{\text{Nul}(A) = \text{Span} \left\{ \begin{bmatrix} 7 \\ -4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -6 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}.}$$

Problem 4

$$A = \begin{bmatrix} 1 & -6 & 4 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}.$$

Row reduce:

$$\begin{bmatrix} 1 & -6 & 4 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -6 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

The equations are

$$x_1 - 6x_2 = 0, \qquad x_3 = 0.$$

Let

$$x_2 = s, \qquad x_4 = t.$$

Then

$$x_1 = 6s, \qquad x_3 = 0.$$

Thus,

$$\mathbf{x} = s \begin{bmatrix} 6 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Therefore,

$$\boxed{\text{Nul}(A) = \text{Span}\left\{ \begin{bmatrix} 6 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.}$$

Problem 5

$$A = \begin{bmatrix} 1 & -2 & 0 & 4 & 0 \\ 0 & 0 & 1 & -9 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

This matrix is already in echelon form. The equations are

$$x_1 - 2x_2 + 4x_4 = 0, \qquad x_3 - 9x_4 = 0, \qquad x_5 = 0.$$

Let

$$x_2 = s, \qquad x_4 = t.$$

Then

$$x_1 = 2s - 4t, \qquad x_3 = 9t, \qquad x_5 = 0.$$

Thus,

$$\mathbf{x} = s \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -4 \\ 0 \\ 9 \\ 1 \\ 0 \end{bmatrix}.$$

Therefore,

$$\boxed{\mathrm{Nul}(A) = \mathrm{Span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 9 \\ 1 \\ 0 \end{bmatrix} \right\}.}$$

Problem 6

$$A = \begin{bmatrix} 1 & 5 & -4 & -3 & 1 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Row reduce:

$$\begin{bmatrix} 1 & 5 & -4 & -3 & 1 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 6 & -8 & 1 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The equations are

$$x_1 + 6x_3 - 8x_4 + x_5 = 0, \qquad x_2 - 2x_3 + x_4 = 0.$$

Let

$$x_3 = s, \qquad x_4 = t, \qquad x_5 = u.$$

Then

$$x_1 = -6s + 8t - u, \qquad x_2 = 2s - t.$$

Therefore,

$$\mathbf{x} = s \begin{bmatrix} -6 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 8 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + u \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Thus,

$$\boxed{\mathrm{Nul}(A) = \mathrm{Span}\left\{ \begin{bmatrix} -6 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 8 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.}$$

Problems 7–14

Problem 7

$$W = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} : a + b + c = 2 \right\}.$$

This set is not a vector space because it does not contain the zero vector. Indeed,

$$0 + 0 + 0 = 0 \neq 2.$$

Therefore,

$$\boxed{W \text{ is not a vector space.}}$$

Problem 8

$$W = \left\{ \begin{bmatrix} r \\ s \\ t \end{bmatrix} : 5r - 1 = s + 2t \right\}.$$

Rewrite the equation as

$$5r - s - 2t = 1.$$

The zero vector does not satisfy this equation because

$$5(0) - 0 - 2(0) = 0 \neq 1.$$

Therefore,

$$\boxed{W \text{ is not a vector space.}}$$

Problem 9

$$W = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} : a - 2b = 4c, \quad 2a = c + 3d \right\}.$$

Rewrite the equations as

$$a - 2b - 4c = 0, \qquad 2a - c - 3d = 0.$$

These are homogeneous linear equations. The solution set of a homogeneous system is always a subspace.

Therefore,

$$\boxed{W \text{ is a vector space.}}$$

Problem 10

$$W = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} : a + 3b = c, \quad b + c + a = d \right\}.$$

Rewrite the equations as

$$a + 3b - c = 0, \qquad a + b + c - d = 0.$$

These are homogeneous linear equations. Therefore, W is the solution set of a homogeneous system.

Thus,

$$\boxed{W \text{ is a vector space.}}$$

Problem 11

$$W = \left\{ \begin{bmatrix} b - 2d \\ 5 + d \\ b + 3d \\ d \end{bmatrix} : b, d \in \mathbb{R} \right\}.$$

This set is not a vector space because it does not contain the zero vector. If the vector were zero, then from the fourth component we would need

$$d = 0.$$

But then the second component becomes

$$5 + d = 5 \neq 0.$$

So the zero vector is not in W.

Therefore,

$$\boxed{W \text{ is not a vector space.}}$$

Problem 12

$$W = \left\{ \begin{bmatrix} b - 5d \\ 2b \\ 2d + 1 \\ d \end{bmatrix} : b, d \in \mathbb{R} \right\}.$$

This set is not a vector space because it does not contain the zero vector. If the fourth component is zero, then

$$d = 0.$$

But then the third component is

$$2d + 1 = 1 \neq 0.$$

Therefore,

$$\boxed{W \text{ is not a vector space.}}$$

Problem 13

$$W = \left\{ \begin{bmatrix} c - 6d \\ d \\ c \end{bmatrix} : c, d \in \mathbb{R} \right\}.$$

Separate the parameters:

$$\begin{bmatrix} c - 6d \\ d \\ c \end{bmatrix} = c \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + d \begin{bmatrix} -6 \\ 1 \\ 0 \end{bmatrix}.$$

Therefore,

$$W = \text{Span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -6 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

Since W is a span,

$$\boxed{W \text{ is a vector space.}}$$

Problem 14

$$W = \left\{ \begin{bmatrix} -a + 2b \\ a - 2b \\ 3a - 6b \end{bmatrix} : a, b \in \mathbb{R} \right\}.$$

Separate the parameters:

$$\begin{bmatrix} -a + 2b \\ a - 2b \\ 3a - 6b \end{bmatrix} = a \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} + b \begin{bmatrix} 2 \\ -2 \\ -6 \end{bmatrix}.$$

But

$$\begin{bmatrix} 2 \\ -2 \\ -6 \end{bmatrix} = -2 \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}.$$

Therefore,

$$W = \text{Span} \left\{ \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} \right\}.$$

Since W is a span,

$$\boxed{W \text{ is a vector space.}}$$

Section 4.3: Linearly Independent Sets; Bases

Problems 1–8

For subsets of $\mathbb{R}^3$, a set is a basis for $\mathbb{R}^3$ exactly when it contains three linearly independent vectors. Equivalently, if the vectors are placed as columns of a matrix, the matrix must have rank 3.

Problem 1

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

Place the vectors as columns:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

This matrix is upper triangular with determinant

$$\det(A) = 1.$$

Therefore, the columns are linearly independent and span $\mathbb{R}^3$.

$$\boxed{\text{This set is a basis for } \mathbb{R}^3.}$$

Problem 2

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

This set contains the zero vector. Any set containing the zero vector is linearly dependent. Also, the rank is only 2, so the set does not span $\mathbb{R}^3$.

$$\boxed{\text{Not a basis; linearly dependent; does not span } \mathbb{R}^3.}$$

Problem 3

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix}, \begin{bmatrix} -3 \\ -5 \\ 1 \end{bmatrix} \right\}.$$

Row reducing the matrix with these columns gives rank 2. Since there are three vectors but rank is not 3, the vectors are linearly dependent and do not span $\mathbb{R}^3$.

$$\boxed{\text{Not a basis; linearly dependent; does not span } \mathbb{R}^3.}$$

Problem 4

$$\left\{ \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} -7 \\ 5 \\ 4 \end{bmatrix} \right\}.$$

The determinant of the matrix with these vectors as columns is

$$-24.$$

Since the determinant is nonzero, the vectors are linearly independent and span $\mathbb{R}^3$.

$$\boxed{\text{This set is a basis for } \mathbb{R}^3.}$$

Problem 5

$$\left\{ \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 9 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -3 \\ 5 \end{bmatrix} \right\}.$$

This set has four vectors in $\mathbb{R}^3$, so it cannot be linearly independent. It also contains the zero vector, so it is definitely linearly dependent.

However, row reduction gives rank 3, so the vectors span $\mathbb{R}^3$.

$$\boxed{\text{Not a basis; linearly dependent; spans } \mathbb{R}^3.}$$

Problem 6

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} -4 \\ -5 \\ 6 \end{bmatrix} \right\}.$$

There are only two vectors in $\mathbb{R}^3$, so they cannot span $\mathbb{R}^3$. Since neither vector is a scalar multiple of the other, they are linearly independent.

$$\boxed{\text{Not a basis; linearly independent; does not span } \mathbb{R}^3.}$$

Problem 7

$$\left\{ \begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ -1 \\ 5 \end{bmatrix} \right\}.$$

There are only two vectors in $\mathbb{R}^3$, so they cannot span $\mathbb{R}^3$. Since neither vector is a scalar multiple of the other, they are linearly independent.

$$\boxed{\text{Not a basis; linearly independent; does not span } \mathbb{R}^3.}$$

Problem 8

$$\left\{ \begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ -5 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -2 \end{bmatrix} \right\}.$$

This set has four vectors in $\mathbb{R}^3$, so it cannot be linearly independent. However, row reducing the matrix with these vectors as columns gives rank 3. Hence the vectors span $\mathbb{R}^3$.

$$\boxed{\text{Not a basis; linearly dependent; spans } \mathbb{R}^3.}$$

Problems 9–10

Problem 9

$$A = \begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 1 & -5 & 4 \\ 3 & -2 & 1 & -2 \end{bmatrix}.$$

Row reducing gives

$$A \sim \begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 1 & -5 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The equations are

$$x_1 - 3x_3 + 2x_4 = 0, \qquad x_2 - 5x_3 + 4x_4 = 0.$$

Let

$$x_3 = s, \qquad x_4 = t.$$

Then

$$x_1 = 3s - 2t, \qquad x_2 = 5s - 4t.$$

Thus,

$$\mathbf{x} = s\begin{bmatrix}3\\5\\1\\0\end{bmatrix} + t\begin{bmatrix}-2\\-4\\0\\1\end{bmatrix}.$$

Therefore,

$$\boxed{\text{Nul}(A) = \text{Span}\left\{\begin{bmatrix}3\\5\\1\\0\end{bmatrix}, \begin{bmatrix}-2\\-4\\0\\1\end{bmatrix}\right\}.}$$

Problem 10

$$A = \begin{bmatrix}1 & 0 & -5 & 1 & 4\\-2 & 1 & 6 & -2 & -2\\0 & 2 & -8 & 1 & 9\end{bmatrix}.$$

Row reducing gives

$$A \sim \begin{bmatrix}1 & 0 & -5 & 0 & 7\\0 & 1 & -4 & 0 & 6\\0 & 0 & 0 & 1 & -3\end{bmatrix}.$$

The equations are

$$x_1 - 5x_3 + 7x_5 = 0, \qquad x_2 - 4x_3 + 6x_5 = 0, \qquad x_4 - 3x_5 = 0.$$

Let

$$x_3 = s, \qquad x_5 = t.$$

Then

$$x_1 = 5s - 7t, \qquad x_2 = 4s - 6t, \qquad x_4 = 3t.$$

Thus,

$$\mathbf{x} = s\begin{bmatrix}5\\4\\1\\0\\0\end{bmatrix} + t\begin{bmatrix}-7\\-6\\0\\3\\1\end{bmatrix}.$$

Therefore,

$$\boxed{\mathrm{Nul}(A) = \mathrm{Span}\left\{ \begin{bmatrix} 5 \\ 4 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ -6 \\ 0 \\ 3 \\ 1 \end{bmatrix} \right\}.}$$

Problems 13–14

Problem 13

Suppose A is row equivalent to

$$B = \begin{bmatrix} 1 & 0 & 6 & 5 \\ 0 & 2 & 5 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where

$$A = \begin{bmatrix} -2 & 4 & -2 & -4 \\ 2 & -6 & -3 & 1 \\ -3 & 8 & 2 & -3 \end{bmatrix}.$$

The pivot columns of B are columns 1 and 2. Therefore, a basis for $\mathrm{Col}(A)$ is given by the corresponding columns of A:

$$\boxed{\mathrm{Col}(A) = \mathrm{Span}\left\{ \begin{bmatrix} -2 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 4 \\ -6 \\ 8 \end{bmatrix} \right\}.}$$

A basis for $\mathrm{Row}(A)$ is given by the nonzero rows of B:

$$\boxed{\mathrm{Row}(A) = \mathrm{Span}\left\{ \begin{bmatrix} 1 & 0 & 6 & 5 \end{bmatrix}, \begin{bmatrix} 0 & 2 & 5 & 3 \end{bmatrix} \right\}.}$$

To find $\mathrm{Nul}(A)$, solve $B\mathbf{x} = \mathbf{0}$:

$$x_1 + 6x_3 + 5x_4 = 0, \qquad 2x_2 + 5x_3 + 3x_4 = 0.$$

Let $x_3 = 2s$ and $x_4 = 2t$ to avoid fractions. Then

$$x_1 = -12s - 10t, \qquad x_2 = -5s - 3t.$$

Thus,

$$\mathbf{x} = s \begin{bmatrix} -12 \\ -5 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} -10 \\ -3 \\ 0 \\ 2 \end{bmatrix}.$$

Therefore,

$$\boxed{\mathrm{Nul}(A) = \mathrm{Span}\left\{ \begin{bmatrix} -12 \\ -5 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -10 \\ -3 \\ 0 \\ 2 \end{bmatrix} \right\}.}$$

Problem 14

Suppose A is row equivalent to

$$B = \begin{bmatrix} 1 & 2 & 0 & 4 & 5 \\ 0 & 0 & 5 & -7 & 8 \\ 0 & 0 & 0 & 0 & -9 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where

$$A = \begin{bmatrix} 1 & 2 & -5 & 11 & -3 \\ 2 & 4 & -5 & 15 & 2 \\ 1 & 2 & 0 & 4 & 5 \\ 3 & 6 & -5 & 19 & -2 \end{bmatrix}.$$

The pivot columns of B are columns 1, 3, and 5. Therefore, a basis for $\mathrm{Col}(A)$ is given by columns 1, 3, and 5 of the original matrix A:

$$\boxed{\mathrm{Col}(A) = \mathrm{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -5 \\ -5 \\ 0 \\ -5 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 5 \\ -2 \end{bmatrix} \right\}.}$$

A basis for $\mathrm{Row}(A)$ is given by the nonzero rows of B:

$$\boxed{\mathrm{Row}(A) = \mathrm{Span}\left\{ \begin{bmatrix} 1 & 2 & 0 & 4 & 5 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 5 & -7 & 8 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & -9 \end{bmatrix} \right\}.}$$

To find $\mathrm{Nul}(A)$, solve $B\mathbf{x} = \mathbf{0}$. The equations are

$$x_1 + 2x_2 + 4x_4 + 5x_5 = 0,$$

$$5x_3 - 7x_4 + 8x_5 = 0,$$

$$-9x_5 = 0.$$

From the third equation,

$$x_5 = 0.$$

Let

$$x_2 = s, \qquad x_4 = 5t.$$

Then

$$x_3 = 7t, \qquad x_1 = -2s - 20t.$$

Thus,

$$\mathbf{x} = s\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -20 \\ 0 \\ 7 \\ 5 \\ 0 \end{bmatrix}.$$

Therefore,

$$\boxed{\text{Nul}(A) = \text{Span}\left\{\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -20 \\ 0 \\ 7 \\ 5 \\ 0 \end{bmatrix}\right\}.}$$

Problems 15–18

For each problem, place the given vectors as columns of a matrix. The pivot columns of that matrix form a basis for the space spanned by the given vectors.

Problem 15

The given vectors are

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -3 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ -3 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -3 \\ -4 \\ 1 \\ 6 \end{bmatrix},$$

$$\mathbf{v}_4 = \begin{bmatrix} 1 \\ -3 \\ -8 \\ 7 \end{bmatrix}, \quad \mathbf{v}_5 = \begin{bmatrix} 2 \\ 1 \\ -6 \\ 9 \end{bmatrix}.$$

Row reduction shows that the pivot columns are columns 1, 2, and 4.
Therefore, a basis for the span is

$$\boxed{\left\{\begin{bmatrix} 1 \\ 0 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ -8 \\ 7 \end{bmatrix}\right\}.}$$

Problem 16

The given vectors are

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -2 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 6 \\ -1 \\ 2 \\ -1 \end{bmatrix},$$

$$\mathbf{v}_4 = \begin{bmatrix} 5 \\ -3 \\ 3 \\ -4 \end{bmatrix}, \quad \mathbf{v}_5 = \begin{bmatrix} 0 \\ 3 \\ -1 \\ 1 \end{bmatrix}.$$

Row reduction shows that the pivot columns are columns 1, 2, and 3. Therefore, a basis for the span is

$$\boxed{\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ -1 \\ 2 \\ -1 \end{bmatrix} \right\}.}$$

Problem 17

The given vectors are

$$\mathbf{v}_1 = \begin{bmatrix} 8 \\ 9 \\ -3 \\ -6 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 4 \\ 5 \\ 1 \\ -4 \\ 4 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -1 \\ -4 \\ -9 \\ 6 \\ -7 \end{bmatrix},$$

$$\mathbf{v}_4 = \begin{bmatrix} 6 \\ 8 \\ 4 \\ -7 \\ 10 \end{bmatrix}, \quad \mathbf{v}_5 = \begin{bmatrix} -1 \\ 4 \\ 11 \\ -8 \\ -7 \end{bmatrix}.$$

Row reduction shows that the pivot columns are columns 1, 2, and 3. Therefore, a basis for the span is

$$\boxed{\left\{ \begin{bmatrix} 8 \\ 9 \\ -3 \\ -6 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 1 \\ -4 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ -4 \\ -9 \\ 6 \\ -7 \end{bmatrix} \right\}.}$$

Problem 18

The given vectors are

$$\mathbf{v}_1 = \begin{bmatrix} -8 \\ 7 \\ 6 \\ 5 \\ -7 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 8 \\ -7 \\ -9 \\ -5 \\ 7 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -8 \\ 7 \\ 4 \\ 5 \\ -7 \end{bmatrix},$$

$$\mathbf{v}_4 = \begin{bmatrix} 1 \\ 4 \\ 9 \\ 6 \\ -7 \end{bmatrix}, \quad \mathbf{v}_5 = \begin{bmatrix} -9 \\ 3 \\ -4 \\ -1 \\ 0 \end{bmatrix}.$$

Row reduction shows that the pivot columns are columns 1, 2, and 4. Therefore, a basis for the span is

$$\boxed{\left\{ \begin{bmatrix} -8 \\ 7 \\ 6 \\ 5 \\ -7 \end{bmatrix}, \begin{bmatrix} 8 \\ -7 \\ -9 \\ -5 \\ 7 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 9 \\ 6 \\ -7 \end{bmatrix} \right\}.}$$

Section 4.4: Coordinate Systems

Problems 1–4

If

$$B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$$

and

$$[\mathbf{x}]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix},$$

then

$$\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n.$$

Problem 1

$$B = \left\{ \begin{bmatrix} 3 \\ -5 \end{bmatrix}, \begin{bmatrix} -4 \\ 6 \end{bmatrix} \right\}, \qquad [\mathbf{x}]_B = \begin{bmatrix} 5 \\ 3 \end{bmatrix}.$$

Thus,

$$\mathbf{x} = 5\begin{bmatrix}3\\-5\end{bmatrix} + 3\begin{bmatrix}-4\\6\end{bmatrix} = \begin{bmatrix}15\\-25\end{bmatrix} + \begin{bmatrix}-12\\18\end{bmatrix} = \begin{bmatrix}3\\-7\end{bmatrix}.$$

$$\boxed{\mathbf{x} = \begin{bmatrix}3\\-7\end{bmatrix}}$$

Problem 2

$$B = \left\{\begin{bmatrix}4\\5\end{bmatrix}, \begin{bmatrix}6\\7\end{bmatrix}\right\}, \qquad [\mathbf{x}]_B = \begin{bmatrix}8\\-5\end{bmatrix}.$$

Thus,

$$\mathbf{x} = 8\begin{bmatrix}4\\5\end{bmatrix} - 5\begin{bmatrix}6\\7\end{bmatrix} = \begin{bmatrix}32\\40\end{bmatrix} + \begin{bmatrix}-30\\-35\end{bmatrix} = \begin{bmatrix}2\\5\end{bmatrix}.$$

$$\boxed{\mathbf{x} = \begin{bmatrix}2\\5\end{bmatrix}}$$

Problem 3

$$B = \left\{\begin{bmatrix}1\\-4\\3\end{bmatrix}, \begin{bmatrix}5\\2\\-2\end{bmatrix}, \begin{bmatrix}4\\-7\\0\end{bmatrix}\right\}, \qquad [\mathbf{x}]_B = \begin{bmatrix}3\\0\\-1\end{bmatrix}.$$

Thus,

$$\mathbf{x} = 3\begin{bmatrix}1\\-4\\3\end{bmatrix} + 0\begin{bmatrix}5\\2\\-2\end{bmatrix} - 1\begin{bmatrix}4\\-7\\0\end{bmatrix}.$$

So

$$\mathbf{x} = \begin{bmatrix}3\\-12\\9\end{bmatrix} + \begin{bmatrix}0\\0\\0\end{bmatrix} + \begin{bmatrix}-4\\7\\0\end{bmatrix} = \begin{bmatrix}-1\\-5\\9\end{bmatrix}.$$

$$\boxed{\mathbf{x} = \begin{bmatrix}-1\\-5\\9\end{bmatrix}}$$

Problem 4

$$B = \left\{ \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -5 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ -7 \\ 3 \end{bmatrix} \right\}, \qquad [\mathbf{x}]_B = \begin{bmatrix} -4 \\ 8 \\ -7 \end{bmatrix}.$$

Thus,

$$\mathbf{x} = -4 \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + 8 \begin{bmatrix} 3 \\ -5 \\ 2 \end{bmatrix} - 7 \begin{bmatrix} 4 \\ -7 \\ 3 \end{bmatrix}.$$

Compute:

$$\mathbf{x} = \begin{bmatrix} 4 \\ -8 \\ 0 \end{bmatrix} + \begin{bmatrix} 24 \\ -40 \\ 16 \end{bmatrix} + \begin{bmatrix} -28 \\ 49 \\ -21 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -5 \end{bmatrix}.$$

$$\boxed{\mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ -5 \end{bmatrix}}$$

Problems 5–8

To find $[\mathbf{x}]_B$, solve

$$\mathbf{x} = c_1 \mathbf{b}_1 + \cdots + c_n \mathbf{b}_n.$$

The coordinate vector is

$$[\mathbf{x}]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}.$$

Problem 5

$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \qquad \mathbf{b}_2 = \begin{bmatrix} 2 \\ -5 \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

Solve

$$c_1 \begin{bmatrix} 1 \\ -3 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ -5 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

This gives

$$c_1 = 8, \qquad c_2 = -5.$$

Therefore,

$$\boxed{[\mathbf{x}]_B = \begin{bmatrix} 8 \\ -5 \end{bmatrix}.}$$

Problem 6

$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \qquad \mathbf{b}_2 = \begin{bmatrix} 5 \\ -6 \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}.$$

Solve

$$c_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2 \begin{bmatrix} 5 \\ -6 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}.$$

The solution is

$$c_1 = -6, \qquad c_2 = 2.$$

Thus,

$$\boxed{[\mathbf{x}]_B = \begin{bmatrix} -6 \\ 2 \end{bmatrix}.}$$

Problem 7

$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} -3 \\ 4 \\ 9 \end{bmatrix}, \quad \mathbf{b}_3 = \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 8 \\ -9 \\ 6 \end{bmatrix}.$$

Solving

$$c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + c_3\mathbf{b}_3 = \mathbf{x}$$

gives

$$c_1 = -1, \qquad c_2 = -1, \qquad c_3 = 3.$$

Therefore,

$$\boxed{[\mathbf{x}]_B = \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix}.}$$

Problem 8

$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 2 \\ 1 \\ 8 \end{bmatrix}, \quad \mathbf{b}_3 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 3 \\ -5 \\ 4 \end{bmatrix}.$$

Solving

$$c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + c_3\mathbf{b}_3 = \mathbf{x}$$

gives

$$c_1 = -2, \qquad c_2 = 0, \qquad c_3 = 5.$$

Therefore,

$$\boxed{[\mathbf{x}]_B = \begin{bmatrix} -2 \\ 0 \\ 5 \end{bmatrix}.}$$

Problems 9–10

The change-of-coordinates matrix from B-coordinates to standard coordinates is

$$P_B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_n \end{bmatrix}.$$

Problem 9

$$B = \left\{ \begin{bmatrix} 2 \\ -9 \end{bmatrix}, \begin{bmatrix} 1 \\ 8 \end{bmatrix} \right\}.$$

Therefore,

$$\boxed{P_B = \begin{bmatrix} 2 & 1 \\ -9 & 8 \end{bmatrix}.}$$

Problem 10

$$B = \left\{ \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -5 \end{bmatrix}, \begin{bmatrix} 8 \\ -2 \\ 7 \end{bmatrix} \right\}.$$

Therefore,

$$\boxed{P_B = \begin{bmatrix} 3 & 2 & 8 \\ -1 & 0 & -2 \\ 4 & -5 & 7 \end{bmatrix}.}$$

Problems 11–12

To find $[\mathbf{x}]_B$ using an inverse matrix, first form

$$P_B = \begin{bmatrix}\mathbf{b}_1 & \mathbf{b}_2\end{bmatrix}.$$

Then

$$\mathbf{x} = P_B[\mathbf{x}]_B,$$

so

$$[\mathbf{x}]_B = P_B^{-1}\mathbf{x}.$$

Problem 11

$$B = \left\{\begin{bmatrix}3\\-5\end{bmatrix}, \begin{bmatrix}-4\\6\end{bmatrix}\right\}, \qquad \mathbf{x} = \begin{bmatrix}2\\-6\end{bmatrix}.$$

Thus,

$$P_B = \begin{bmatrix}3 & -4\\-5 & 6\end{bmatrix}.$$

Compute

$$P_B^{-1} = \begin{bmatrix}-3 & -2\\-\frac{5}{2} & -\frac{3}{2}\end{bmatrix}.$$

Therefore,

$$[\mathbf{x}]_B = P_B^{-1}\mathbf{x} = \begin{bmatrix}-3 & -2\\-\frac{5}{2} & -\frac{3}{2}\end{bmatrix}\begin{bmatrix}2\\-6\end{bmatrix} = \begin{bmatrix}6\\4\end{bmatrix}.$$

Thus,

$$\boxed{[\mathbf{x}]_B = \begin{bmatrix}6\\4\end{bmatrix}.}$$

Problem 12

$$B = \left\{\begin{bmatrix}4\\5\end{bmatrix}, \begin{bmatrix}6\\7\end{bmatrix}\right\}, \qquad \mathbf{x} = \begin{bmatrix}2\\0\end{bmatrix}.$$

Thus,

$$P_B = \begin{bmatrix}4 & 6\\5 & 7\end{bmatrix}.$$

Compute

$$P_B^{-1} = \begin{bmatrix}-\frac{7}{2} & 3\\\frac{5}{2} & -2\end{bmatrix}.$$

Therefore,

$$[\mathbf{x}]_B = P_B^{-1}\mathbf{x} = \begin{bmatrix} -\frac{7}{2} & 3 \\ \frac{5}{2} & -2 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -7 \\ 5 \end{bmatrix}.$$

Thus,

$$\boxed{[\mathbf{x}]_B = \begin{bmatrix} -7 \\ 5 \end{bmatrix}.}$$

Copyright