

Detailed Solutions to Quiz #4

MATH 031 Applied Linear Algebra

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Question 1

Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix}.$$

(a) Is

$$u = (-2, 1, 4, -3)$$

in $\text{Nul}(A)$? Is it in $\text{Col}(A)$?

(b) Is

$$v = (3, 5)$$

in $\text{Col}(A)$? Is it in $\text{Nul}(A)$?

Important Definitions

The **null space** of A , written $\text{Nul}(A)$, is the set of all vectors x such that

$$Ax = 0.$$

Since A is a 2×4 matrix, it takes vectors from \mathbb{R}^4 and outputs vectors in \mathbb{R}^2 . Therefore,

$$\text{Nul}(A) \subseteq \mathbb{R}^4.$$

The **column space** of A , written $\text{Col}(A)$, is the span of the columns of A . Since each column of A is a vector in \mathbb{R}^2 , we have

$$\text{Col}(A) \subseteq \mathbb{R}^2.$$

The columns of A are

$$a_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \quad a_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad a_3 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad a_4 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}.$$

Part (a): Checking whether $u \in \text{Nul}(A)$

We are given

$$u = \begin{bmatrix} -2 \\ 1 \\ 4 \\ -3 \end{bmatrix}.$$

To check whether $u \in \text{Nul}(A)$, we compute Au :

$$Au = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 4 \\ -3 \end{bmatrix}.$$

Now compute each entry.

For the first row:

$$1(-2) + 2(1) + 3(4) + 4(-3) = -2 + 2 + 12 - 12 = 0.$$

For the second row:

$$4(-2) + 3(1) + 2(4) + 1(-3) = -8 + 3 + 8 - 3 = 0.$$

Therefore,

$$Au = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Since $Au = 0$, we conclude that

$$\boxed{u \in \text{Nul}(A)}.$$

Part (a): Checking whether $u \in \text{Col}(A)$

The vector u is a vector in \mathbb{R}^4 :

$$u = \begin{bmatrix} -2 \\ 1 \\ 4 \\ -3 \end{bmatrix} \in \mathbb{R}^4.$$

However, the column space of A is a subspace of \mathbb{R}^2 , because the columns of A each have two entries:

$$\text{Col}(A) \subseteq \mathbb{R}^2.$$

Since $u \in \mathbb{R}^4$ but $\text{Col}(A) \subseteq \mathbb{R}^2$, the vector u cannot be in $\text{Col}(A)$.

Thus,

$$\boxed{u \notin \text{Col}(A)}.$$

Part (b): Checking whether $v \in \text{Col}(A)$

We are given

$$v = \begin{bmatrix} 3 \\ 5 \end{bmatrix}.$$

Since $v \in \mathbb{R}^2$ and $\text{Col}(A) \subseteq \mathbb{R}^2$, it makes sense to ask whether v is in the column space of A .

To check this, we ask whether there exists a vector

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

such that

$$Ax = v.$$

That is,

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}.$$

One way to show that $v \in \text{Col}(A)$ is to write v as a linear combination of the columns of A .

We only need two columns if those two columns already span \mathbb{R}^2 . Consider the first two columns:

$$a_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \quad a_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

We want constants c_1 and c_2 such that

$$c_1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}.$$

This gives the system

$$\begin{cases} c_1 + 2c_2 = 3, \\ 4c_1 + 3c_2 = 5. \end{cases}$$

From the first equation,

$$c_1 = 3 - 2c_2.$$

Substitute this into the second equation:

$$4(3 - 2c_2) + 3c_2 = 5.$$

Simplify:

$$12 - 8c_2 + 3c_2 = 5,$$

$$12 - 5c_2 = 5,$$

$$\begin{aligned} -5c_2 &= -7, \\ c_2 &= \frac{7}{5}. \end{aligned}$$

Then

$$c_1 = 3 - 2\left(\frac{7}{5}\right) = 3 - \frac{14}{5} = \frac{15}{5} - \frac{14}{5} = \frac{1}{5}.$$

Therefore,

$$v = \frac{1}{5} \begin{bmatrix} 1 \\ 4 \end{bmatrix} + \frac{7}{5} \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

Equivalently,

$$A \begin{bmatrix} \frac{1}{5} \\ \frac{7}{5} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}.$$

Thus v is a linear combination of the columns of A , so

$$\boxed{v \in \text{Col}(A)}.$$

Part (b): Checking whether $v \in \text{Nul}(A)$

The null space of A is a subset of \mathbb{R}^4 :

$$\text{Nul}(A) \subseteq \mathbb{R}^4.$$

However,

$$v = \begin{bmatrix} 3 \\ 5 \end{bmatrix} \in \mathbb{R}^2.$$

Since v is not even the correct size to be multiplied by A , the expression Av is not defined. Therefore, v cannot be in the null space of A .

Thus,

$$\boxed{v \notin \text{Nul}(A)}.$$

Final Answers for Question 1

$$\boxed{u \in \text{Nul}(A)}$$

$$\boxed{u \notin \text{Col}(A)}$$

$$\boxed{v \in \text{Col}(A)}$$

$$\boxed{v \notin \text{Nul}(A)}$$

Question 2

Consider the vector space P_2 of real polynomials of degree 2 or less, and the shifted basis

$$\mathcal{B} = \{1, t - 1, (t - 1)^2\}.$$

Find

$$[1 + t^2]_{\mathcal{B}}.$$

That is, we want to find the coordinates of the polynomial

$$1 + t^2$$

relative to the basis

$$\mathcal{B} = \{1, t - 1, (t - 1)^2\}.$$

Understanding the Problem

The coordinate vector

$$[1 + t^2]_{\mathcal{B}}$$

means that we want to write the polynomial $1 + t^2$ as a linear combination of the basis elements in \mathcal{B} .

So we want constants c_1 , c_2 , and c_3 such that

$$1 + t^2 = c_1(1) + c_2(t - 1) + c_3(t - 1)^2.$$

Once we find c_1 , c_2 , and c_3 , the coordinate vector will be

$$[1 + t^2]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}.$$

Method 1: Expanding and Matching Coefficients

Start with

$$1 + t^2 = c_1 + c_2(t - 1) + c_3(t - 1)^2.$$

Now expand each term:

$$c_2(t - 1) = c_2t - c_2,$$

and

$$c_3(t - 1)^2 = c_3(t^2 - 2t + 1).$$

Therefore,

$$1 + t^2 = c_1 + c_2t - c_2 + c_3(t^2 - 2t + 1).$$

Distribute c_3 :

$$1 + t^2 = c_1 + c_2t - c_2 + c_3t^2 - 2c_3t + c_3.$$

Group like terms:

$$1 + t^2 = c_3t^2 + (c_2 - 2c_3)t + (c_1 - c_2 + c_3).$$

Now compare this with

$$1 + t^2 = 1 + 0t + 1t^2.$$

Matching coefficients gives the system

$$\begin{cases} c_3 = 1, \\ c_2 - 2c_3 = 0, \\ c_1 - c_2 + c_3 = 1. \end{cases}$$

From the first equation,

$$c_3 = 1.$$

Substitute $c_3 = 1$ into the second equation:

$$c_2 - 2(1) = 0.$$

Thus,

$$c_2 - 2 = 0,$$

so

$$c_2 = 2.$$

Now substitute $c_2 = 2$ and $c_3 = 1$ into the third equation:

$$c_1 - 2 + 1 = 1.$$

Simplify:

$$c_1 - 1 = 1.$$

Therefore,

$$c_1 = 2.$$

So we have

$$c_1 = 2, \quad c_2 = 2, \quad c_3 = 1.$$

Thus,

$$1 + t^2 = 2(1) + 2(t - 1) + 1(t - 1)^2.$$

Therefore,

$$\boxed{[1 + t^2]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}}.$$

Method 2: Using the Substitution $t = (t - 1) + 1$

Another way to see the answer is to rewrite t in terms of $(t - 1)$:

$$t = (t - 1) + 1.$$

Then

$$t^2 = ((t - 1) + 1)^2.$$

Using the formula $(a + b)^2 = a^2 + 2ab + b^2$, we get

$$t^2 = (t - 1)^2 + 2(t - 1) + 1.$$

Therefore,

$$1 + t^2 = 1 + [(t - 1)^2 + 2(t - 1) + 1].$$

Simplify:

$$1 + t^2 = 2 + 2(t - 1) + (t - 1)^2.$$

Thus, relative to the basis

$$\mathcal{B} = \{1, t - 1, (t - 1)^2\},$$

the coefficients are

$$2, \quad 2, \quad 1.$$

So again,

$$\boxed{[1 + t^2]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}}.$$

Final Answers**Question 1**

$$\boxed{u \in \text{Nul}(A)}$$

$$\boxed{u \notin \text{Col}(A)}$$

$$\boxed{v \in \text{Col}(A)}$$

$$\boxed{v \notin \text{Nul}(A)}$$

Question 2

$$\boxed{[1 + t^2]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}}$$

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