Detailed Solutions to Quiz #5

Diagonalization, Eigenvalues, and Eigenvectors

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Throughout this solution, we use the characteristic polynomial convention

$$p_A(\lambda) = \det(\lambda I - A).$$

Question 1

Determine whether the matrix

$$A = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}$$

is diagonalizable. If yes, find a matrix P such that

$$D = P^{-1}AP$$

is diagonal.

Step 1: Find the characteristic polynomial

We compute

$$p_A(\lambda) = \det(\lambda I - A).$$

Since

$$\lambda I = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix},$$

we have

$$\lambda I - A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} \lambda - 4 & 1 \\ -2 & \lambda - 1 \end{bmatrix}.$$

Therefore,

$$p_A(\lambda) = \det \begin{bmatrix} \lambda - 4 & 1 \\ -2 & \lambda - 1 \end{bmatrix}.$$

Using the determinant formula for a 2×2 matrix,

$$p_A(\lambda) = (\lambda - 4)(\lambda - 1) - 1(-2).$$

Thus,

$$p_A(\lambda) = (\lambda - 4)(\lambda - 1) + 2.$$

Expanding,

$$(\lambda - 4)(\lambda - 1) = \lambda^2 - 5\lambda + 4.$$

Therefore,

$$p_A(\lambda) = \lambda^2 - 5\lambda + 4 + 2 = \lambda^2 - 5\lambda + 6.$$

Factoring,

$$p_A(\lambda) = (\lambda - 2)(\lambda - 3).$$

Thus, the eigenvalues are

$$\boxed{\lambda = 2} \qquad \text{and} \qquad \boxed{\lambda = 3}.$$

Step 2: Decide whether A is diagonalizable

The matrix A is a 2×2 matrix and has two distinct eigenvalues:

$$\lambda = 2 \qquad \text{and} \qquad \lambda = 3.$$

A square matrix with n distinct eigenvalues is diagonalizable. Since this 2×2 matrix has two distinct eigenvalues, it is diagonalizable.

$$\boxed{A \text{ is diagonalizable.}}$$

Step 3: Find an eigenvector for $\lambda = 2$

We solve

$$(A - 2I)x = 0.$$

First,

$$A - 2I = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix}.$$

So we solve

$$\begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

This gives the equation

$$2x - y = 0.$$

Therefore,

$$y = 2x.$$

Let $x = 1$. Then $y = 2$. Thus, an eigenvector corresponding to $\lambda = 2$ is

$$v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Step 4: Find an eigenvector for $\lambda = 3$

We solve

$$(A - 3I)x = 0.$$

First,

$$A - 3I = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix}.$$

So we solve

$$\begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

This gives the equation

$$x - y = 0.$$

Therefore,

$$x = y.$$

Let $x = 1$. Then $y = 1$. Thus, an eigenvector corresponding to $\lambda = 3$ is

$$v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Step 5: Construct P and D

To diagonalize A, we place the eigenvectors as columns of P. Since

$$v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{corresponds to} \quad \lambda = 2,$$

and

$$v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{corresponds to} \quad \lambda = 3,$$

we choose

$$P = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}.$$

Then the corresponding diagonal matrix is

$$D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}.$$

Thus,

$$\boxed{P = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}} \qquad \text{and} \qquad \boxed{D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}.}$$

Therefore,

$$\boxed{D = P^{-1}AP.}$$

Check

We can verify using the relationship

$$AP = PD.$$

Compute

$$AP = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}.$$

Thus,

$$AP = \begin{bmatrix} 4(1) + (-1)(2) & 4(1) + (-1)(1) \\ 2(1) + 1(2) & 2(1) + 1(1) \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 4 & 3 \end{bmatrix}.$$

Now compute

$$PD = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}.$$

So

$$PD = \begin{bmatrix} 2 & 3 \\ 4 & 3 \end{bmatrix}.$$

Hence,

$$AP = PD.$$

This confirms that

$$D = P^{-1}AP.$$

Question 2

Determine whether the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

is diagonalizable. If yes, find a matrix P such that

$$D = P^{-1}AP$$

is diagonal. If not, explain why.

Step 1: Find the characteristic polynomial

The matrix A is upper triangular. For an upper triangular matrix, the eigenvalues are the diagonal entries. Therefore, the eigenvalues are

$$\lambda = 1, \quad \lambda = 1, \quad \lambda = 2.$$

So the characteristic polynomial is

$$p_A(\lambda) = (\lambda - 1)^2(\lambda - 2).$$

Thus,

$$\boxed{p_A(\lambda) = (\lambda - 1)^2(\lambda - 2).}$$

The eigenvalue $\lambda = 1$ has algebraic multiplicity 2, and the eigenvalue $\lambda = 2$ has algebraic multiplicity 1.

Step 2: Find the eigenspace for $\lambda = 1$

We solve

$$(A - I)x = 0.$$

First,

$$A - I = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

Now solve

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

This gives the equations

$$y = 0,$$
$$2z = 0,$$

and

$$z = 0.$$

Therefore,

$$y = 0 \qquad \text{and} \qquad z = 0.$$

The variable x is free. Thus,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Therefore,

$$E_1 = \operatorname{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

So

$$\dim(E_1) = 1.$$

Step 3: check using the eigenvalue $\lambda = 2$

We solve

$$(A - 2I)x = 0.$$

First,

$$A - 2I = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Now solve

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

This gives the equations

$$-x + y = 0,$$

and

$$-y + 2z = 0.$$

From the first equation,

$$y = x.$$

From the second equation,

$$y = 2z.$$

Thus,

$$x = y = 2z.$$

Let $z = 1$. Then

$$x = 2 \qquad \text{and} \qquad y = 2.$$

So an eigenvector for $\lambda = 2$ is

$$v = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}.$$

Hence,

$$E_2 = \text{span} \left\{ \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \right\}.$$

The eigenspace for $\lambda = 1$ contributes only one linearly independent eigenvector, and the eigenspace for $\lambda = 2$ contributes one linearly independent eigenvector. So we only get

$$1 + 1 = 2$$

linearly independent eigenvectors total.

Since a 3×3 matrix needs 3 linearly independent eigenvectors to be diagonalizable, this matrix is not diagonalizable.

$$\boxed{A \text{ is not diagonalizable.}}$$

Final Answer for Question 2

$$\boxed{A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \text{ is not diagonalizable because it does not have enough linearly independent eigenvectors.}}$$

More specifically,

$$\boxed{\dim(E_1) = 1 \quad \text{but} \quad \lambda = 1 \text{ has algebraic multiplicity 2.}}$$

Final Summary

Question	Eigenvalues	Diagonalizable?
1	$2, 3$	Yes
2	$1, 1, 2$	No

For Question 1,

$$\boxed{P = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}, \qquad D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}}$$

and

$$\boxed{D = P^{-1}AP.}$$

For Question 2, no such diagonal matrix $D = P^{-1}AP$ exists because the matrix is not diagonalizable.